

On The Multilinear Fractional Integral Operators With Correlation Kernels

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Abstract

In this paper, we study a class of multilinear fractional integral operators which have correlation kernels $\prod_{1 \leq i < j \leq k} |x_i - x_j|^{-\alpha_{ij}}$. The necessary and sufficient conditions are obtained under which these operators are bounded from $L^{p_1} \times \cdots \times L^{p_k}$ into L^q . As a consequence, we also get the endpoint estimates from $L^{p_1} \times \cdots \times L^{p_k}$ to BMO of these operators.

Keywords: Multilinear fractional integral operators, Correlation kernels, Hardy-Littlewood-Sobolev inequality, Selberg integrals.

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1 Introduction

Fractional integral operators arise frequently in various subjects such as Fourier analysis and partial differential equations. The Riesz potential is a classical fractional integral operator which was extended to various multilinear cases by many authors; see [5], [1], [8], [16], [10], [11], [2], [23] and [3].

In this paper, we mainly study mapping properties of the multilinear fractional integral operators with correlation kernels of the form $\prod |x_i - x_j|^{-\alpha_{ij}}$. These operators can be written as

$$T(f_1, \dots, f_k)(x_{k+1}) = \int_{\mathbb{R}^{nk}} \frac{\prod_{i=1}^k f_i(x_i)}{\prod_{1 \leq i < j \leq k+1} |x_i - x_j|^{\alpha_{ij}}} dx_1 dx_2 \cdots dx_k, \quad (1.1)$$

with $f_i \in C_0^\infty(\mathbb{R}^n)$ and $\alpha_{ij} \geq 0$. It is clear that T reduces to Riesz potentials when $k = 1$. The meaning of the definition of T can be given by the distribution theory. It is natural to assume that the kernel of T is a Schwartz kernel such that it maps $(C_0^\infty(\mathbb{R}^n))^k$ into $\mathcal{D}'(\mathbb{R}^n)$ boundedly. This assumption require some restrictions on the exponents α_{ij} . Another issue is to determine the necessary and sufficient conditions under which T has a bounded extension from $L^{p_1} \times \cdots \times L^{p_k}$ into L^q . More precisely, we shall establish the following inequality

$$\|T(f_1, f_2, \dots, f_k)\|_{L^q} \leq C \prod_{i=1}^k \|f_i\|_{L^{p_i}} \quad (1.2)$$

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with the constant C independent of $f_i \in L^{p_i}$. The Hardy-Littlewood-Sobolev inequality is a special case of the above inequality for $k = 1$. For $k \geq 2$, it is more convenient to write T as a multilinear functional of the form

$$\Lambda(f_1, f_2, \dots, f_{k+1}) = \int_{\mathbb{R}^{n(k+1)}} \frac{\prod_{i=1}^{k+1} f_i(x_i)}{\prod_{1 \leq i < j \leq k+1} |x_i - x_j|^{\alpha_{ij}}} dx_1 dx_2 \cdots dx_{k+1}. \quad (1.3)$$

Then the boundedness of T is equivalent to

$$|\Lambda(f_1, f_2, \dots, f_{k+1})| \leq C \prod_{i=1}^{k+1} \|f_i\|_{L^{p_i}}. \quad (1.4)$$

The problems discussed above have close relation with several topics in Fourier analysis. In [5], Christ applied a special case of the inequality (1.4) to establish the endpoint estimates of the restriction of the Fourier transform to curves in higher dimensions. Beckner stated a conformally invariant inequality of the form (1.4) in [1]. Morpurgo obtained sharp inequalities for trace functionals of pseudo-differential operators on the sphere S^n and multilinear fractional integrals appear explicitly in the calculation of zeta functions of those operators; see [17]. Those sharp inequalities obtained in [17] also relies on the strict rearrangement of a class of functionals with kernels $\prod K_{ij}(|x_i - x_j|)$. In this paper we shall give the necessary and sufficient conditions under which the multilinear functional Λ is bounded. In this direction, Wu obtained partial results in his dissertation [23]. One of our main results in this paper can be stated as follows.

Theorem 1.1 *Let Λ be the multilinear functional defined by (1.3) with all $\alpha_{ij} \geq 0$. Assume $1 < p_i < \infty$ for $1 \leq i \leq k+1$. Then the inequality (1.4) is valid for all $f_i \in L^{p_i}(\mathbb{R}^n)$ if and only if the following three conditions hold simultaneously.*

- (i) $\sum_{i=1}^{k+1} \frac{1}{p_i} + \sum_{1 \leq i < j \leq k+1} \frac{\alpha_{ij}}{n} = k+1$;
 - (ii) $\sum_I \frac{\alpha_{ij}}{n} < |I| - 1$ for $I \subset \{1, 2, \dots, k+1\}$ with $|I| \geq 2$;
 - (iii) For all nonempty $I \subsetneq \{1, 2, \dots, k+1\}$, one of the following two statements is true:
 - (a) $\sum_I \frac{1}{p_i} + \sum_I \frac{\alpha_{ij}}{n} < |I|$;
 - (b) $\sum_I \frac{1}{p_i} + \sum_I \frac{\alpha_{ij}}{n} = |I|$, $\sum_{I^c} \frac{1}{p_i} \geq 1$ and $\sum_J \left(\frac{1}{p_i} + \sum_{u \in I} \frac{\alpha_{iu}}{n} \right) + \sum_J \frac{\alpha_{ij}}{n} \leq |J|$
- for all subsets J of I^c .

Here we use the notations $\alpha_{ij} = \alpha_{ji}$ and $\alpha_{ii} = 0$ for $1 \leq i, j \leq k+1$. The cardinality of I is denoted by $|I|$ and I^c is the complement of I . The above summations are defined by

$$\sum_I \frac{\alpha_{ij}}{n} = \sum_{i, j \in I; i < j} \frac{\alpha_{ij}}{n} \quad \text{and} \quad \sum_I \frac{1}{p_i} = \sum_{i \in I} \frac{1}{p_i}$$

for all subsets I of $\{1, 2, \dots, k+1\}$.

Some remarks will help clarify the necessity of conditions in the theorem. The equality (i) is easily verified by a dilation argument. The system of inequalities (ii) ensure that T has a Schwartz kernel, i.e. T is a bounded mapping from the product test function space into the distribution space. In §6, we shall see that the inequalities (ii) is the necessary and sufficient condition ensuring $\int_{(S^n)^k} \prod |\xi_i - \xi_j|^{-\alpha_{ij}} d\sigma(\xi_1) d\sigma(\xi_2) \cdots d\sigma(\xi_k) < \infty$ which appears explicitly in the formula of the sharp constant of Λ in the conformally invariant setting; see [1] and [9] and also [7] for its connections with the Selberg integral. There are some previously known results related to the theorem. In [10], Grafakos and Kalton obtained partial results for $k = 2$ by multilinear interpolation with some further assumptions on the exponents α_{ij} . When α_{ij} and p_i are two constants, the result is the same as Proposition 2.2 in [5]. Stein and Weiss considered a class of weighted Hardy-Littlewood-Sobolev inequalities which are known as the Stein-Weiss potentials in the literature. This potentials is generalized to the multilinear case in the theorem when (b) of the condition (iii) holds for some proper subset J of $\{1, 2, \dots, k+1\}$.

Now we review some basic properties of Riesz potentials; see Stein [20]. For $0 < \alpha < n$, Riesz potentials I_α are defined by

$$I_\alpha(f)(x) = \frac{\Gamma(n/2 - \alpha/2)}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy$$

for $f \in C_0^\infty$. The Hardy-Littlewood-Sobolev inequality states

$$\|I_\alpha(f)\|_q \leq C \|f\|_p, \quad 1 < p < q < \infty, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

This estimate is due to Hardy and Littlewood in [12] and [13] for $n = 1$. Sobolev [19] obtained this inequality for general n .

We also study the endpoint boundedness of the functional Λ . In [21], Stein and Weiss proved that I_α is also bounded from H^1 to $L^{n/(n-\alpha)}$. By duality, it follows that I_α has a bounded extension from $L^{n/\alpha}$ to BMO . We shall prove that this is also true for the multilinear operator T . This result is contained in the following theorem.

Theorem 1.2 *Let T be a multilinear operator of the form (1.1) with all $\alpha_{ij} \geq 0$. Assume $p_{k+1} \in \{1, \infty\}$ and $1 < p_i < \infty$ for $1 \leq i \leq k$. Suppose $\{\alpha_{ij}\}$ and $\{p_i\}$ satisfy the conditions (i) and (ii) in Theorem 1.1 and the first type (a) of the condition (iii):*

$$\sum_I \frac{1}{p_i} + \sum_I \frac{\alpha_{ij}}{n} < |I|$$

for any nonempty and proper subset I of $\{1, 2, \dots, k+1\}$ unless $I = \{k+1\}$ when $p_{k+1} = 1$. Then we have the following estimates:

$$\begin{aligned} \|T(f_1, f_2, \dots, f_k)\|_{L^1} &\leq C \prod_{i=1}^k \|f_i\|_{p_i} & \text{if } p_{k+1} = \infty \\ \text{and } \|T(f_1, f_2, \dots, f_k)\|_{BMO} &\leq C \prod_{i=1}^k \|f_i\|_{p_i} & \text{if } p_{k+1} = 1 \end{aligned}$$

with $f_i \in C_0^\infty$ and both constants C independent of choices of f_i .

Concerning notation, the parameters $\{\alpha_{ij}\}$ are defined to be symmetric. In other words, we assume $\alpha_{ij} = \alpha_{ji}$ for all $1 \leq i, j \leq k+1$ and all $\alpha_{ii} = 0$. For any given subset J of $\{1, 2, \dots, k+1\}$, we use the summation conventions $\sum_J 1/p_i$ and $\sum_J \alpha_{ij}$ to denote $\sum_{i \in J} 1/p_i$ and $\sum_{i < j; i, j \in J} \alpha_{ij}$, respectively. If J consists of a single point, we set $\sum_J \alpha_{ij} = 0$. This convention is also extended to general parameters $\{\gamma_i\}$ and symmetric $\{\beta_{ij}\}$. The constant C means a positive number which may vary from place to place. For $A, B \geq 0$, $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$. By this notation, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. We use $A \wedge B$ to denote $\min\{A, B\}$ and $|J|$ to denote the cardinality of a index set J . For a measurable set E in \mathbb{R}^n , $|E|$ is its Lebesgue measure. For two sets E and F , $E - F$ means $E \cap F^c$ where F^c is the complement of F . For brevity, we use $S = \{1, 2, \dots, k\}$ throughout the paper.

The present paper is organized as follows. The section 2 contains some basic lemmas and previously known results which will be used in subsequent sections. The necessity of the conditions (i), (ii) and (iii) in Theorem 1.1 is proved in §3. In §4, we treat the trilinear functional as a model case. In §5, we shall give a useful L^∞ estimate and prove that the condition (ii) in Theorem 1.1 is sufficient for the local integrability of T . A complete proof of our main results is presented in §6.

2 Preliminaries

In this section, we shall establish basic estimates for certain fractional integrals and present some useful results which are previously known. The structure of the $k+1$ -point correlation kernel in (1.1) suggests that we can apply a general rearrangement lemma to reduce the matters to Selberg integrals (1.3) with $f_i(x) = |x|^{-n/p_i}$. As a result, we are able to prove Theorem 1.1 by estimating a class of explicit integrals. These ideas will be completed in section 6. The general rearrangement lemma can be stated as follows; see [4].

Definition 2.1 Assume f is a measurable function in \mathbb{R}^n . Let $\lambda_f(s) = |\{x : |f(x)| > s\}|$, $s > 0$, be the distribution of f relative to the Lebesgue measure. Suppose $\lambda_f(s) < \infty$ for some $s < \infty$. If f^* is another function in \mathbb{R}^n with the following properties:

- (i) $f^*(x_1) = f^*(x_2)$ if $|x_1| = |x_2|$;
- (ii) $f^*(x_2) \leq f^*(x_1)$ if $|x_1| \leq |x_2|$;
- (iii) $\lambda_f(s) = \lambda_{f^*}(s)$ for $s > 0$;

then f^* is called the symmetric decreasing rearrangement of f in \mathbb{R}^n .

By this definition, we can state the general rearrangement inequality as follows.

Lemma 2.1 Suppose that each f_i is a measurable function on \mathbb{R}^n for $1 \leq i \leq k$. Let f_i^* be the symmetric decreasing rearrangement of f_i in \mathbb{R}^n . For $m \geq 1$ and real numbers $\{a_{ij}\}_{k \times m}$, we have

$$\int_{\mathbb{R}^{nm}} \prod_{i=1}^k \left| f_i \left(\sum_{j=1}^m a_{ij} x_j \right) \right| dx_1 \cdots dx_m \leq \int_{\mathbb{R}^{nm}} \prod_{i=1}^k \left| f_i^* \left(\sum_{j=1}^m a_{ij} x_j \right) \right| dx_1 \cdots dx_m.$$

We refer the reader to [4] for its proof. By this lemma, we see that $|\Lambda(f_1, \dots, f_{k+1})|$ is not greater than $\Lambda(f_1^*, \dots, f_{k+1}^*)$. This rearrangement inequality together with the simple observation $f^*(x) \leq \omega_n^{-1/p} \|f\|_p |x|^{-n/p}$ for $f \in L^p$ justifies our consideration of the integral (1.3) with $f_i(x_i) = |x|^{-n/p_i}$, where we use ω_n to denote the Lebesgue measure of the unit ball in \mathbb{R}^n . The following multilinear interpolation theorem makes these ideas possible.

For $k + 1$ points x_1, x_2, \dots, x_{k+1} in \mathbb{R}^k , we call these points affinely independent if they do not lie in a hyperplane in \mathbb{R}^k simultaneously.

Theorem 2.2 *Assume that T is a k -linear operator which is bounded from $L^{p_{1j},1} \times \dots \times L^{p_{kj},1}$ into $L^{q_j,\infty}$ for $0 < p_{ij} \leq \infty$ and $0 < q_j \leq \infty$ with $1 \leq i \leq k$ and $1 \leq j \leq k + 1$. Assume also that the $k + 1$ points $(1/p_{1j}, \dots, 1/p_{kj})$ are affinely independent in \mathbb{R}^k . If there are $k + 1$ real numbers λ_i with positive $\lambda_1, \dots, \lambda_k$ such that*

$$\frac{1}{q_j} = \sum_{i=1}^k \frac{\lambda_i}{p_{ij}} + \lambda_{k+1}, \quad 1 \leq j \leq k + 1,$$

then T is bounded from $L^{p_1,t_1} \times \dots \times L^{p_k,t_k}$ into $L^{q,t}$ with $(1/p_1, \dots, 1/p_k, 1/q)$ lying in the open convex hull of $k + 1$ points $(1/p_{1j}, \dots, 1/p_{kj}, 1/q_j)$ in \mathbb{R}^{k+1} and $0 < t_i, t \leq \infty$ satisfying

$$\sum_{i=1}^k \frac{1}{t_i} \geq \frac{1}{t}.$$

This theorem was previously known; see [14]. It makes our reduction of Theorem 1.1 to a class of special integrals possible. We also refer the reader to a similar variant called the multilinear Marcinkiewicz interpolation in [10].

The L^1 estimate in Theorem 1.2 implies that $\Lambda(f_1, \dots, f_{k+1})$ is bounded by a constant multiple of $\prod_{i=1}^{k+1} \|f_i\|_{p_i}$ with $p_{k+1} = \infty$. Let $f_{k+1} \equiv 1$. We see that the integral in (1.1) with respect to x_{k+1} is a generalization of the so called beta integral with $k = 2$. An induction argument requires that upper bounds of the integral are of the form $\prod_S |x_i - x_j|^{-\beta_{ij}}$ with suitable parameters $\{\beta_{ij}\}$. In other words, we need estimates of the following form

$$\int_{\mathbb{R}^n} \prod_{i=1}^k |x_i - x_{k+1}|^{-\alpha_{i,k+1}} dx_{k+1} \leq C \prod_S |x_i - x_j|^{-\beta_{ij}}. \quad (2.5)$$

The following theorem gives us desired estimates.

Theorem 2.3 *Assume $\alpha_1, \alpha_2, \dots, \alpha_k$ satisfy $0 < \alpha_i < n$. If $\sum_{i=1}^k \alpha_i > n$, then the following estimate*

$$\int_{\mathbb{R}^n} \prod_{i=1}^k |t - x_i|^{-\alpha_i} dt \leq C \sum_{u=1}^k L_u(x_1, x_2, \dots, x_k)$$

holds for arbitrary x_1, x_2, \dots, x_k in \mathbb{R}^n with

$$L_u(x_1, x_2, \dots, x_k) = \begin{cases} d_S^{n-\sum_S \alpha_i} \left(\chi_{\{\sum_{S-\{u\}} \alpha_i < n\}} + \chi_{\{\sum_{S-\{u\}} \alpha_i = n\}} \log \frac{2d_S}{d_{S-\{u\}}} \right) \\ d_S^{-\alpha_u} \int_{\mathbb{R}^n} \prod_{S-\{u\}} |t - x_i|^{-\alpha_i} dt, & \text{if } \sum_{S-\{u\}} \alpha_i > n \end{cases}$$

with the characteristic function χ taken relative to $\alpha_1, \dots, \alpha_k$. Here $S = \{1, 2, \dots, k\}$ and $d_I = \sum_I |x_i - x_j|$ for subsets I of $S = \{1, \dots, k\}$ with $|I| \geq 2$.

Remark 2.1 *There are some explicit formulas concerning the integral in the theorem. These formulas take the form*

$$\int_{\mathbb{R}^n} \prod_{i=1}^k |t - x_i|^{-\alpha_i} dt = C \prod_{1 \leq i < j \leq k} |x_i - x_j|^{-\gamma_{ij}}$$

with a constant C independent of x_i . When $k = 2$, this is just the n -dimension version of the beta integral formula; see Stein [20]. For $k = 3$, Grafakos and Morpurgo ([9]) proved the equality with $\gamma_{ij} = \alpha_i + \alpha_j - n$ when $\alpha_1 + \alpha_2 + \alpha_3 = 2n$. However, its generalization to other cases is impossible. Recently, Wu and Yan have proved that the above equality cannot be true in the remaining cases (i) $\alpha_1 + \alpha_2 + \alpha_3 \neq 2n$ when $k = 3$ and (ii) $k \geq 4$.

Lemma 2.4 Let $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive numbers satisfying $\sum_{i=1}^k \alpha_i = n$. For k points x_1, x_2, \dots, x_k in the unit ball $B_1(0) \subseteq \mathbb{R}^n$, it is true that

$$\int_{|t| \leq 2} \prod_{i=1}^k |t - x_i|^{-\alpha_i} dt \leq C \log \frac{C}{d_S}, \quad (2.6)$$

where C depends on dimension $n, \alpha_1, \dots, \alpha_k$. Moreover, the reverse inequality is also true.

Proof. Assume $|x_1 - x_k| = \max_S |x_i - x_j|$. Then d_S is comparable to $|x_1 - x_k|$. By dilation,

$$\int_{|t| \leq 2d_S} \prod_{i=1}^k |t - (x_i - x_1)|^{-\alpha_i} dt \leq C(n, \alpha_1, \dots, \alpha_k).$$

We may further assume $d_S \leq 1$. Actually, when $d_S > 1$ the integral in the lemma has lower and upper bounds which depend only on n and α_i . On the other hand, it is easy to see that

$$\begin{aligned} \int_{2d_S \leq |t| \leq 3} \prod_{i=1}^k |t - (x_i - x_1)|^{-\alpha_i} dt &\approx \int_{2d_S \leq |t| \leq 3} |t|^{-n} dt \\ &= C \log \frac{3}{2d_S} \end{aligned}$$

which also implies the reverse inequality. Thus we conclude the proof. \square

We now turn to the proof of Theorem 2.3.

Proof. Without loss of generality, we may assume $|x_1 - x_k| = \max_S |x_i - x_j|$. We shall estimate the integral over $B_{d_S/2}(x_1)$ and its complement separately. We first observe that

$$\int_{B_{d_S/2}(x_1)} \prod_{i=1}^k |t - x_i|^{-\alpha_i} dt \leq C d_S^{-\alpha_k} \int_{B_{d_S/2}(0)} \prod_{i=1}^{k-1} |t - x_i + x_1|^{-\alpha_i} dt.$$

We may apply Lemma 2.4 to obtain that the integral of $\prod_{i=1}^{k-1} |t - x_i + x_1|^{-\alpha_i}$ over $B_{d_S/2}(0)$ is bounded by a constant multiple of $d_S^{n - \sum_{i=1}^{k-1} \alpha_i} \left(\chi_{\{\sum_{S-\{k\}} \alpha_i < n\}} + \chi_{\{\sum_{S-\{k\}} \alpha_i = n\}} \log \frac{2d_S}{d_{S-\{k\}}} \right)$ if $\sum_{i=1}^{k-1} \alpha_i \leq n$. For $\sum_{i=1}^{k-1} \alpha_i > n$, it is clear that

$$\int_{B_{d_S/2}(0)} \prod_{i=1}^{k-1} |t - x_i + x_1|^{-\alpha_i} dt \leq \int_{\mathbb{R}^n} \prod_{S-\{k\}} |t - x_i|^{-\alpha_i} dt.$$

Now we treat the integral outside the ball $B_{d_S/2}(x_1)$. It is easy to see that

$$\begin{aligned} \int_{B_{d_S/2}^c(x_1)} \prod_{i=1}^k |t - x_i|^{-\alpha_i} dt &\leq C \int_{d_S/2 \leq |t| \leq 2d_S} \prod_{i=1}^k |t - (x_i - x_1)|^{-\alpha_i} dt \\ &\leq C d_S^{-\alpha_1} \int_{|t| \leq 2d_S} \prod_{i=2}^k |t - (x_i - x_1)|^{-\alpha_i} dt. \end{aligned}$$

The integral of $\prod_{i=2}^k |t - (x_i - x_1)|^{-\alpha_i}$ over $|t| \leq 2d_S$ can be treated similarly as above. Combing above estimates, we conclude the integral is bounded by a constant multiple of $L_1(x_1, \dots, x_k) + L_k(x_1, \dots, x_k)$. This completes the proof of the theorem. \square

We shall see that the proof of Theorem 1.1 is closely related to the existence of solutions to a system of linear inequalities. A system of linear inequalities in \mathbb{R}^n is given by

$$(II.1) \begin{cases} f_i(x) = (v_i, x) < a_i, & 1 \leq i \leq m \\ f_i(x) = (v_i, x) \leq a_i, & m+1 \leq i \leq k \end{cases}$$

where $v_i \in \mathbb{R}^n$, $a_i \in \mathbb{R}$ and (\cdot, \cdot) denotes the standard inner product in \mathbb{R}^n . It is worthwhile noting that we may incorporate an linear equality into a system of linear inequalities. Indeed, we may write $g(x) = (v, x) = a$ as an equivalent system of two linear inequalities given by $g(x) \leq a$ and $-g(x) \leq -a$.

Lemma 2.5 *Suppose that the system $f_i(x) = (v_i, x) \leq a_i$ for $m+1 \leq i \leq k$ has at least one solution. Then there exists a solution $x \in \mathbb{R}^n$ to the system (II.1) if and only if*

$$\sum_{i=1}^k \lambda_i a_i > 0$$

for all nonnegative numbers λ_i satisfying $\sum_{i=1}^k \lambda_i f_i = 0$ with at least one $\lambda_i > 0$ for $1 \leq i \leq m$.

This lemma is a special case of the existence theorem of systems of convex inequalities in \mathbb{R}^n . However, it can be proved by a simple method using the concept of elementary vectors of an subspace of \mathbb{R}^n . We refer the reader to §22 (Page 198) in [18] by Rockafellar; see also [6] for its extensions to general vector spaces.

3 Necessity

In this section, we shall prove the necessity of conditions (i), (ii) and (iii) in Theorem 1.1. Indeed Wu gave a proof in his thesis [23], we present the details here for convenience of the reader.

Assume that the inequality (1.4) is true with a constant C independent of f_i . We replace f_i by its dilation $\delta_\lambda(f_i)(x) = f_i(\lambda x)$ for $\lambda > 0$. By a change of variables, we see that (i) must hold by letting $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$.

To show the necessity of (ii), we take all $f_i = \chi_{B_1(0)}$. We shall replace $\{1, 2, \dots, k+1\}$ by $S = \{1, 2, \dots, k\}$. We claim that if there were a subset $J \subset S$ with $|J| \geq 2$ such that $\sum_J \alpha_{ij} \geq (|J| - 1)n$, we would obtain $\int_{(B_1(0))^{|J|}} \prod_J |x_i - x_j|^{-\alpha_{ij}} dV_J = \infty$, where $B_1(0)$ is the unit ball centered at the origin in \mathbb{R}^n and dV_J is the product Lebesgue measure $\prod_J dx_i$. The argument is essentially the same for different J 's. Assume $J = S$. Then $\int_{(B_1(0))^k} \prod_S |x_i - x_j|^{-\alpha_{ij}} dV_S$ is equal to

$$\begin{aligned} & \int_{B_1(0)} \left(\int_{(B_1(x_1))^{k-1}} \prod_{i=2}^k |x_i|^{-\alpha_{1i}} \prod_{2 \leq i < j \leq k} |x_i - x_j|^{-\alpha_{ij}} dx_2 \cdots dx_k \right) dx_1 \\ & \geq C \int_{(B_{1/2}(0))^{k-1}} \prod_{i=2}^k |x_i|^{-\alpha_{1i}} \prod_{2 \leq i < j \leq k} |x_i - x_j|^{-\alpha_{ij}} dx_2 \cdots dx_k. \end{aligned}$$

Write $X = (x_2, \dots, x_k) \in \mathbb{R}^{n(k-1)}$. Let $B_r(0_m)$ be the unit ball centered at the origin in \mathbb{R}^m with radius $r > 0$. It is clear that $B_{1/2}(0_{n(k-1)})$ is contained in $(B_{1/2}(0_n))^{k-1}$. We may regard the integrand $\prod_{i=2}^k |x_i|^{-\alpha_{1i}} \prod_{2 \leq i < j \leq k} |x_i - x_j|^{-\alpha_{ij}}$ as a homogeneous function of degree $-\sum_S \alpha_{ij}$ in $\mathbb{R}^{n(k-1)}$. Its integral over a ball centered at the origin in $\mathbb{R}^{n(k-1)}$ is infinite since its order of homogeneity is less than or equal to $-(k-1)n$.

It remains to prove the necessity of (iii). We first prove that for any $J \subset S$ with $|J| \geq 2$,

$$\sum_J \alpha_{ij} + \sum_J \frac{n}{p_i} \leq |J|n. \quad (3.7)$$

Assume the converse holds, i.e., there exists some $J_0 \subset S$ with at least two elements and the above inequality is not true. We choose $0 < \lambda_i < n/p_i$ for each $i \in J_0$, such that

$$\sum_{J_0} \alpha_{ij} + \sum_{J_0} \lambda_i = |J_0|n.$$

Let $f_i(y) = \chi_{\{|y| \leq 1\}} |y|^{-\lambda_i}$ for each $i \in J_0$ and f_i be the characteristic function of the unit ball $B_1(0)$ for $i \notin J_0$. It follows that

$$\Lambda(f_1, \dots, f_{k+1}) \geq C \int_{(B_1(0))^{|J_0|}} \prod_{J_0} |x_i|^{-\lambda_i} \prod_{J_0} |x_i - x_j|^{-\alpha_{ij}} dV_{J_0} = \infty$$

as pointed out above.

If for some proper subset J of $\{1, 2, \dots, k+1\}$ with at least two elements, the inequality (3.7) became an inequality, we would have that $\sum_{J^c} 1/p_i$ is not less than 1. Assume $\sum_{J^c} 1/p_i < 1$. We can choose $0 < \lambda_i < p_i$ such that $\sum_{J^c} 1/\lambda_i < 1$. Let $f_i(y) = |y|^{-n/p_i} \chi_{\{|y| > 2\}} (\log |y|)^{-1/\lambda_i}$ for each $i \in J^c$ and f_i be the characteristic function of the unit ball centered at the origin for $i \in J$. Substituting these functions into the functional Λ , we have

$$\begin{aligned} \Lambda(f_1, \dots, f_{k+1}) &\geq C \int_{(B_2^c(0))^{|J^c|}} \prod_{J^c} |f_i(x_i)| \prod_{J^c} |x_i|^{-\beta_i} \prod_{J^c} |x_i - x_j|^{-\alpha_{ij}} dV_{J^c} \\ &= C \int_{(B_2^c(0))^{|J^c|}} \prod_{J^c} |x_i|^{-n/p_i} (\log |x_i|)^{-1/\lambda_i} \prod_{J^c} |x_i|^{-\beta_i} \prod_{J^c} |x_i - x_j|^{-\alpha_{ij}} dV_{J^c}, \end{aligned} \quad (3.8)$$

where β_i is equal to $\sum_{u \in J} \alpha_{iu}$ for each $i \in J^c$.

Since the conditions (i), (ii) and (iii) in Theorem 1.1 are invariant under the permutation group on $k+1$ letters, we may assume $J^c = \{1, 2, \dots, l\}$ with $1 \leq l \leq k$. If $l = 1$, it follows immediately from the fact $n/p_1 + \beta_1 = n$ that the right side integral in (3.8) is infinite since $1/\lambda_1 < 1$. Now we treat the case $l > 1$. Replacing the region $(B_2^c(0))^l$ by its proper subset Ω_l consisting of all points (x_1, \dots, x_l) such that $x_1 \in B_2^c(0)$ and $|x_i| \geq 2|x_{i-1}|$ for $2 \leq i \leq l$, we obtain

$$\int_{|x_l| \geq 2|x_{l-1}|} |x_l|^{-n/p_l} (\log |x_l|)^{-1/\lambda_l} |x_l|^{-\beta_l} |x_l|^{-\sum_{i=1}^{l-1} \alpha_{il}} dx_l \geq C |x_{l-1}|^{-\xi_l} (\log |x_{l-1}|)^{-1/\lambda_l},$$

where $\beta_l = \sum_{i=l+1}^{k+1} \alpha_{il}$ and $\xi_l = n/p_l + \sum_{i=1}^{k+1} \alpha_{il} - n$. Substituting this estimate into the integral in (3.8), we see that $\Lambda(f_1, \dots, f_{k+1})$ is not less than a constant multiple of

$$\int_{\Omega_{l-1}} \prod_{i=1}^{l-1} |x_i|^{-n/p_i - \delta_i^{(l-1)} \xi_l} (\log |x_i|)^{-1/\lambda_i - \delta_i^{(l-1)}/\lambda_l} \prod_{i=1}^{l-1} |x_i|^{-\beta_i} \prod_{1 \leq i < j \leq l-1} |x_i - x_j|^{-\alpha_{ij}} dx_1 \cdots dx_{l-1},$$

where Ω_{l-1} is the region $x_1 \in B_2^c(0)$ and $|x_i| \geq 2|x_{i-1}|$ for $2 \leq i \leq l-1$, δ_i^j equals one if $i = j$ and zero otherwise. Continuing the process with $l-1$ steps, we obtain the resulting estimate

$$\Lambda(f_1, \dots, f_{k+1}) \geq C \int_{|x_1| \geq 2} |x_1|^{-\xi_1} (\log |x_1|)^{-\sum_{i=1}^l 1/\lambda_i} dx_1$$

with

$$\xi_1 = \sum_{i=1}^l \frac{n}{p_i} + \sum_{1 \leq i < j \leq l} \alpha_{ij} + \sum_{i=1}^l \sum_{j=l+1}^{k+1} \alpha_{ij} - (l-1)n = n$$

by the condition (i). Recall that $\sum_{i=1}^l 1/\lambda_i$ is less than 1. The above integral is infinite. This contradicts the boundedness of Λ . Hence $\sum_{j \in J^c} 1/p_j \geq 1$.

We remains to show that certain additional requirements are necessary in Theorem 1.1 when (iii) contains equalities for some proper subsets of $\{1, 2, \dots, k+1\}$.

Theorem 3.1 *Assume $1 \leq p_i \leq \infty$ and $\alpha_{ij} \geq 0$ for $1 \leq i, j \leq k+1$. Suppose the multilinear functional Λ given by (1.3) satisfies*

$$|\Lambda(f_1, f_2, \dots, f_{k+1})| \leq C \prod_{i=1}^{k+1} \|f_i\|_{p_i}$$

with C independent of f_i . If J_0 is a nonempty proper subset of $\{1, 2, \dots, k+1\}$ satisfying

$$\sum_{J_0} \frac{1}{p_i} + \sum_{J_0} \frac{\alpha_{ij}}{n} = |J_0|,$$

then we have

$$\int_{(\mathbb{R}^n)^{|J_0|}} \prod_{J_0} |f_i(x_i)| \prod_{J_0} |x_i - x_j|^{-\alpha_{ij}} dV_{J_0} \leq C \prod_{J_0} \|f_i\|_{p_i} \quad (3.9)$$

$$\text{and } \int_{(\mathbb{R}^n)^{|J_0^c|}} \prod_{J_0^c} |f_i(x_i)| \prod_{J_0^c} |x_i - x_j|^{-\alpha_{ij}} \prod_{J_0^c} |x_i|^{-\beta_i} dV_{J_0^c} \leq C \prod_{J_0^c} \|f_i\|_{p_i} \quad (3.10)$$

with $\beta_i = \sum_{u \in J_0} \alpha_{iu}$ for each $i \in J_0^c$ and both constants C independent of f_i . Moreover, it is also true that

$$\sum_J \left(\frac{1}{p_i} + \frac{\beta_i}{n} \right) + \sum_J \frac{\alpha_{ij}}{n} \leq |J| \quad (3.11)$$

for all nonempty subsets J of J_0^c .

Proof. For each $\epsilon > 0$, let $f_i = \epsilon^{-n/p_i} \chi_{\{|y| < \epsilon/2\}}$ for each $i \in J_0$. By $\sum_{J_0} 1/p_i + \sum_{J_0} \alpha_{ij}/n = |J_0|$, we have

$$\int_{(\mathbb{R}^n)^{|J_0|}} \prod_{J_0} |f_i(x_i)| \prod_{J_0} |x_i - x_j|^{-\alpha_{ij}} dV_{J_0} \geq C,$$

where C is a constant depending only on n and $|J_0|$ but not on ϵ . For given nonnegative $f_i \in L^{p_i}$ with $i \in J_0^c$, it follows from the boundedness of Λ that

$$\int_{(B_\epsilon^c(0))^{|J_0^c|}} \prod_{J_0^c} |f_i(x_i)| \prod_{J_0^c} |x_i - x_j|^{-\alpha_{ij}} \prod_{J_0^c} |x_i|^{-\beta_i} dV_{J_0^c} \leq C \prod_{J_0^c} \|f_i\|_{p_i}$$

which is just desired by letting $\epsilon \rightarrow 0$. The first inequality (3.9) can be obtained by a similar argument. Indeed, put $f_i = \epsilon^{-n/p_i} \chi_{\{\epsilon < |y| < 2\epsilon\}}$ for each $i \in J_0^c$. For nonnegative functions f_i with $i \in J_0$, we also have

$$\int_{(\mathbb{R}^n)^{|J_0^c|}} \prod_{J_0^c} |f_i(x_i)| \prod_{J_0^c} |x_i - x_j|^{-\alpha_{ij}} \left(\prod_{i \in J_0^c} \prod_{j \in J_0} |x_i - x_j|^{-\alpha_{ij}} \right) dV_{J_0^c} \geq C$$

for $|x_j| < \epsilon/2$ with $j \in J_0$, where the constant C is independent of ϵ and $x_j \in B_{\epsilon/2}(0)$ with $j \in J_0$. Similarly, we then obtain

$$\int_{(B_{\epsilon/2}(0))^{|J_0|}} \prod_{J_0} |f_i(x_i)| \prod_{J_0} |x_i - x_j|^{-\alpha_{ij}} dV_{J_0} \leq C \prod_{J_0} \|f_i\|_{p_i}$$

where $f_i \in L^{p_i}$ with $i \in J_0$ and the constant C is independent of f_i and ϵ . By letting $\epsilon \rightarrow \infty$, the desired inequality follows. The inequalities (3.11) can be proved similarly as (3.7) by invoking (3.10). We omit the details here. \square

Combining above results, the proof of the necessity part of Theorem 1.1 is complete.

4 The Model Case of Trilinear Functionals

To clarify the issue, we shall prove our main results in the case $k = 2$ which serves as a model. The argument in this section provides an illuminating insight into the ideas to be used in subsequent sections. There is no need to utilize results related to the existence of solutions to a system of linear inequalities since parameters involved are easy to deal with.

In this section, we mainly consider the following trilinear functional,

$$\Lambda(f_1, f_2, f_3) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_1(x_1) f_2(x_2) f_3(x_3)}{|x_1 - x_2|^{\alpha_{12}} |x_1 - x_3|^{\alpha_{13}} |x_2 - x_3|^{\alpha_{23}}} dx_1 dx_2 dx_3 \quad (4.12)$$

where f_1, f_2 and f_3 are initially assumed to be smooth with compact support and $\alpha_{ij} \geq 0$. We remark that for $k = 2$ the condition (iii) in Theorem 1.1 can be simplified. Indeed, suppose that for $J = \{2, 3\}$ it is true that

$$\sum_J \frac{n}{p_i} + \sum_J \alpha_{ij} = \frac{n}{p_2} + \frac{n}{p_3} + \alpha_{23} = 2n.$$

Then by Theorem 3.1, we would obtain

$$\left| \int_{\mathbb{R}^n} f_1(x_1) |x_1|^{-\alpha_{12}-\alpha_{13}} dx_1 \right| \leq C \|f\|_{p_1}, \quad f \in C_0^\infty.$$

This implies $p_1 = 1$ and $\alpha_{12} = \alpha_{13} = 0$ since α_{12} and α_{13} are assumed to be nonnegative. Thus the absolute convergent of $\Lambda(f_1, f_2, f_3)$ is equivalent to the boundedness of Riesz potentials, i.e. the Hardy-Littlewood-Sobolev inequality. We are not interested in these trivial cases. For this reason, conditions in Theorem 1.1 for $k = 2$ can be written as follows,

$$(IV) \begin{cases} (i) & \sum_{i=1}^3 n/p_i + \sum_{1 \leq i < j \leq 3} \alpha_{ij} = 3n; \\ (ii) & \sum_J \alpha_{ij} < (|J| - 1)n, \quad \text{for } J \subset \{1, 2, 3\} \text{ with } |J| \geq 2; \\ (iii) & \sum_J n/p_i + \sum_J \alpha_{ij} < |J|n \quad \text{for } J \subset \{1, 2, 3\} \text{ with } 1 \leq |J| \leq 2. \end{cases}$$

In the statement of Theorem 1.1, we also assume $1 < p_i < \infty$ for $i = 1, 2, 3$. However, it has been also pointed out in the introduction that Theorem 1.1 also holds if only one p_i takes 1 or ∞ with L^1 replaced by H^1 under certain assumptions. We shall establish these endpoint estimates on conditions (i), (ii) and (iii) of (IV) in this section.

First assume that $p_1 = \infty$ and $1 < p_2, p_3 < \infty$ satisfy the system of inequalities (IV). By (i) and (iii) for $J = \{2, 3\}$, it is easy to see that $\alpha_{12} + \alpha_{13} > n$. Then it follows that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_1(x_1)f_2(x_2)f_3(x_3)}{|x_1 - x_2|^{\alpha_{12}}|x_1 - x_3|^{\alpha_{13}}|x_2 - x_3|^{\alpha_{23}}} dx_1 dx_2 dx_3 \right| \\ & \leq C \|f_1\|_{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_2(x_2)f_3(x_3)|}{|x_2 - x_3|^{\alpha_{12} + \alpha_{13} + \alpha_{23} - n}} dx_2 dx_3 \\ & \leq C \|f_1\|_{\infty} \|f_2\|_{p_2} \|f_3\|_{p_3} \end{aligned}$$

where the Hardy-Littlewood-Sobolev inequality has been used in the last inequality. Thus the L^{∞} estimates have been set up. To obtain the BMO estimate in Theorem 1.2, we shall first establish Theorem 1.1 in the case $k = 2$.

Theorem 4.1 *Assume $\{p_i\}$ and $\{\alpha_{ij}\}$ are given as in Theorem 1.1 with $k = 2$ satisfying the system (IV) of linear inequalities. Then we have*

$$\int_{\mathbb{R}^{2n}} \prod_{1 \leq i < j \leq 3} |x_i - x_j|^{-\alpha_{ij}} |x_2|^{-n/p_2} |x_3|^{-n/p_3} dx_2 dx_3 \leq C |x_1|^{-n(1-1/p_1)} \quad (4.13)$$

with C independent of x_1 .

Proof. Observe that $\alpha_{13} + \alpha_{23} + n/p_3 > n$. If either α_{13} or α_{23} were zero, the desired inequality would follow immediately. Assume now both α_{13} and α_{23} are positive. Then by Lemma 2.3 we have

$$\int_{\mathbb{R}^n} |x_1 - x_3|^{-\alpha_{13}} |x_2 - x_3|^{-\alpha_{23}} |x_3|^{-n/p_3} dx_3 \leq C \sum_{i=1}^3 \mathbf{L}_i(x_1, x_2).$$

By inserting each \mathbf{L}_i to (4.13) in place of the integral with respect to x_3 , we shall show that (4.13) also holds. We first treat the estimate involving \mathbf{L}_1 . In other words, it is true that

$$\int_{\mathbb{R}^n} |x_1 - x_2|^{-\alpha_{12}} |x_2|^{-n/p_2} \mathbf{L}_1(x_1, x_2) dx_2 \leq C |x_1|^{-n(1-1/p_1)}. \quad (4.14)$$

Case 1: $\alpha_{23} + n/p_3 < n$.

By Theorem 2.3, \mathbf{L}_1 equals $(|x_1 - x_2| + |x_1| + |x_2|)^{n-\alpha_{13}-\alpha_{23}-n/p_3}$ in this case. We can distribute the power $\alpha_{13} + \alpha_{23} + n/p_3 - n$ into that of two terms $|x_1 - x_2|^{-\alpha_{12}}$ and $|x_2|^{-n/p_2}$ appropriately. We claim that there exist nonnegative numbers δ_1 , δ_2 and δ_{12} such that $\alpha_{12} + \delta_{12}$, $1/p_1 + \delta_1$ and $1/p_2 + \delta_2$ satisfy the assumptions in Theorem 1.1 with $k = 1$. In other words, these parameters satisfy the following system of inequalities:

$$\begin{cases} (i) & \delta_{12} + n(\delta_1 + \delta_2) = \alpha_{13} + \alpha_{23} + n/p_3 - n; \\ (ii) & 0 \leq \delta_{12} < n - \alpha_{12}, \delta_1 \geq 0, \delta_2 \geq 0; \\ (iii) & \delta_1 < 1 - 1/p_1, \delta_2 < 1 - 1/p_2. \end{cases}$$

Put

$$\gamma = (\alpha_{13} + \alpha_{23} + n/p_3 - n)/(\alpha_{13} + \alpha_{23} + n/p_3).$$

We may take $\delta_{12} = \gamma(n - \alpha_{12})$, $\delta_1 = \gamma(1 - 1/p_1)$ and $\delta_2 = \gamma(1 - 1/p_2)$. It is easily verified that such choices are legitimate. Then all matters have been reduced to the simplest case $k = 1$. Indeed, the integral on the left side of (4.14) is less than or equal to

$$|x_1|^{-n\delta_1} \int_{\mathbb{R}^n} |x_1 - x_2|^{-\alpha_{12} - \delta_{12}} |x_2|^{-n/p_2 - n\delta_2} dx_2 \leq C|x_1|^{-n(1-1/p_1)}$$

which is just desired. It should be mentioned that the choice of δ_{12} , δ_1 and δ_2 implies $\alpha_{12} + \delta_{12} + n/p_2 + n\delta_2 > n$.

Case 2: $\alpha_{23} + n/p_3 = n$.

In this case \mathbf{L}_1 equals

$$(|x_1 - x_2| + |x_1| + |x_2|)^{-\alpha_{13}} \log\{2(|x_1 - x_2| + |x_1| + |x_2|)/|x_2|\}.$$

A slight modification is needed. Let δ_1 , δ_2 and δ_{12} be a solution of the system in the first case. Put $\bar{\delta}_{12} = \delta_{12} - \varepsilon$, $\bar{\delta}_1 = \delta_1$ and $\bar{\delta}_2 = \delta_2 + \varepsilon/n$. Then $\bar{\delta}_{12}$, $\bar{\delta}_1$ and $\bar{\delta}_2$ are just desired with sufficiently small $\varepsilon > 0$.

Case 3: $\alpha_{23} + n/p_3 > n$.

We shall reduce the estimate (4.13) to the case $\alpha_{13} = 0$ in which (4.13) is easily verified. For this purpose, we require nonnegative numbers δ_1 , δ_2 and δ_{12} such that $\{1/p_1 + \delta_1, 1/p_2 + \delta_2, \alpha_{12} + \delta_{12}\}$ also satisfies the system (IV). In other words, we must have

$$\begin{cases} (i) & \delta_{12} + n(\delta_1 + \delta_2) = \alpha_{13}, \delta_{12} \geq 0, \delta_1 \geq 0, \delta_2 \geq 0; \\ (ii) & \delta_{12} < n - \alpha_{12}, \delta_1 < 1 - 1/p_1, \delta_2 < 1 - 1/p_2; \\ (iii) & \frac{n}{p_1} + \frac{n}{p_2} + (\alpha_{12} + \delta_{12}) + n(\delta_1 + \delta_2) < 2n; \\ (iv) & \frac{n}{p_1} + \frac{n}{p_3} + n\delta_1 < 2n, \quad \frac{n}{p_2} + \frac{n}{p_3} + n\delta_2 + \alpha_{23} < 2n. \end{cases}$$

By the assumption $\alpha_{23} + n/p_3 > n$ it follows that $n/p_1 + n/p_3 + \alpha_{12} + \alpha_{13} < 2n$. Hence it is easy to see that (iii) and the first inequality of (iv) are redundant and can be left out. Let $\gamma = \alpha_{13}/(\alpha_{13} + n)$. We can take $\delta_{12} = \gamma(n - \alpha_{12})$, $\delta_1 = \gamma(1 - 1/p_1)$ and $\delta_2 = \gamma(2 - 1/p_2 - 1/p_3 - \alpha_{23}/n)$. Then $\{\delta_{12}, \delta_1, \delta_2\}$ is a solution of the above system. Then the integral in (4.14) equals a constant multiple of

$$\begin{aligned} & \int_{\mathbb{R}^n} |x_1 - x_2|^{-\alpha_{12}} |x_2|^{-n/p_2} (|x_1 - x_2| + |x_1| + |x_2|)^{-\alpha_{13}} |x_2|^{-(\alpha_{23} + n/p_3 - n)} dx_2 \\ & \leq |x_1|^{-n\delta_1} \int_{\mathbb{R}^n} |x_1 - x_2|^{-\alpha_{12} - \delta_{12}} |x_2|^{-n/p_2 - n\delta_2 - (\alpha_{23} + n/p_3 - n)} dx_2 \\ & \leq C|x_1|^{-n(1-1/p_1)}, \end{aligned}$$

where we have used the fact

$$\alpha_{12} + \delta_{12} + n/p_2 + n/p_3 + n\delta_2 + \alpha_{23} - n = 2n - n/p_1 - n\delta_1 > n.$$

Combing above results, we have obtained (4.14). Similarly, (4.14) also holds with $\mathbf{L}_1(x_1, x_2)$ replaced by $\mathbf{L}_2(x_1, x_2)$. Now we turn to the corresponding estimate involving \mathbf{L}_3 . In the case $\alpha_{13} + \alpha_{23} < n$, the treatment is the same as that of \mathbf{L}_1 in the case $\alpha_{23} + n/p_3 < n$. For the case $\alpha_{13} + \alpha_{23} = n$, the treatment is also similar. We shall consider the main case $\alpha_{13} + \alpha_{23} > n$. We have then

$$\mathbf{L}_3(x_1, x_2) = \left(|x_1 - x_2| + |x_1| + |x_2|\right)^{-n/p_3} |x_1 - x_2|^{n - \alpha_{13} - \alpha_{23}}.$$

Then we shall solve the following system of linear inequalities:

$$\begin{cases} (i) & \delta_{12} + n(\delta_1 + \delta_2) = n/p_3, \delta_{12} \geq 0, \delta_1 \geq 0, \delta_2 \geq 0; \\ (ii) & \delta_{12} < 2n - \alpha_{12} - \alpha_{13} - \alpha_{23}; \\ (iii) & \delta_1 < 1 - 1/p_1, \delta_2 < 1 - 1/p_2. \end{cases}$$

Take $\gamma = 1/(1+p_3)$. Let $\delta_{12} = \gamma(2n - \alpha_{12} - \alpha_{13} - \alpha_{23})$, $\delta_1 = \gamma(1 - 1/p_1)$ and $\delta_2 = \gamma(1 - 1/p_2)$ for which above inequalities are satisfied. Then we see that the integral in (4.14) with \mathbf{L}_3 replacing \mathbf{L}_1 equals a constant multiple of

$$\begin{aligned} & \int_{\mathbb{R}^n} |x_1 - x_2|^{-\alpha_{12}} |x_2|^{-n/p_2} (|x_1 - x_2| + |x_1| + |x_2|)^{-n/p_3} |x_1 - x_2|^{-(\alpha_{13} + \alpha_{23} - n)} dx_2 \\ & \leq |x_1|^{-n\delta_1} \int_{\mathbb{R}^n} |x_1 - x_2|^{-\alpha_{12} - \delta_{12} - (\alpha_{13} + \alpha_{23} - n)} |x_2|^{-n/p_2 - n\delta_2} dx_2 \\ & \leq C|x_1|^{-n(1-1/p_1)}. \end{aligned}$$

The proof of Theorem 4.1 is complete. \square

Now we can prove the boundedness of T in (1.1) in the case $k = 2$ under assumptions given by the system (IV). With $\{\alpha_{ij}\}$ fixed, if (p_1, p_2, p_3) satisfies the system (IV) and $\varepsilon > 0$ is sufficiently small, we see that if $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ satisfies $|1/\bar{p}_i - 1/p_i| < \varepsilon$ and $(1/\bar{p}_1, 1/\bar{p}_2, 1/\bar{p}_3)$ lies in the hyperplane in \mathbb{R}^3

$$x_1 + x_2 + x_3 = 3 - (\alpha_{12} + \alpha_{13} + \alpha_{23})/n,$$

then $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ also satisfies (iii) in the system (IV). For this reason, we can choose three points $(p_1^{(i)}, p_2^{(i)}, p_3^{(i)})$ for $1 \leq i \leq 3$ near (p_1, p_2, p_3) in the above hyperplane such that $(p_2^{(i)}, p_3^{(i)})$ are affinely independent in \mathbb{R}^2 and (p_2, p_3) lies in the open convex hull of $\{(p_2^{(i)}, p_3^{(i)})\}$. By Theorem 4.1, we see that T is bounded from $L^{p_2^{(i)}} \times L^{p_3^{(i)}}$ into $L^{q_1^{(i)}, \infty}$ with $q_1^{(i)}$ being the conjugate exponent of $p_1^{(i)}$. In fact, this statement can be proved by a duality argument. For $f_2 \in L^{p_2^{(i)}}$, $f_3 \in L^{p_3^{(i)}}$ and $g \in L^{p_1^{(i)}, 1}$, we apply the rearrangement inequality in §2 to obtain $|\Lambda(g, f_2, f_3)| \leq \Lambda(g^*, f_2^*, f_3^*)$. This implies $|\Lambda(g, f_2, f_3)| \leq C\|g\|_{L^{p_1^{(i)}, 1}} \|f_2\|_{L^{p_2^{(i)}}} \|f_3\|_{L^{p_3^{(i)}}}$. Hence the boundedness of T follows. Observe that

$$1/q_1^{(i)} = 1/p_2^{(i)} + 1/p_3^{(i)} - (2 - \alpha_{12}/n - \alpha_{13}/n - \alpha_{23}/n)$$

for $1 \leq i \leq 3$ and assumptions in the system (IV) imply $1/p_2 + 1/p_3 \geq 1/p_1'$ where p_1' is conjugate to p_1 . It follows from the interpolation result of Theorem 2.2 that T is bounded from $L^{p_2} \times L^{p_3}$ into $L^{p_1'}$. Thus we have completed the proof of Theorem 1.1 in the case $k = 2$. Recall we have obtained the L^1 estimate of T in Theorem 1.2 with $p_3 = \infty$ at the beginning of this section. Therefore it remains to prove the *BMO* estimate of T .

For $p_1 = 1$, we want to establish the statement that the trilinear functional Λ in (4.12) is bounded on $H^1 \times L^{p_2} \times L^{p_3}$. It is convenient to study the following bilinear operator,

$$T(f_2, f_3)(x_1) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_2(x_2)f_3(x_3)}{\prod_{1 \leq i < j \leq 3} |x_i - x_j|^{\alpha_{ij}}} dx_2 dx_3 \quad (4.15)$$

for $f_2, f_3 \in C_0^\infty$. Assume $1 < p_2, p_3 < \infty$ and $\alpha_{ij} \geq 0$ satisfy the system of inequalities (IV) except the inequality (iii) for $J = \{1\}$. By the duality between H^1 and BMO , it is enough to show that T has a bounded extension from $L^{p_2} \times L^{p_3}$ into BMO , i.e.,

$$\|T(f_2, f_3)\|_{BMO} \leq C \|f_2\|_{p_2} \|f_3\|_{p_3} \quad (4.16)$$

with C independent of $f_2, f_3 \in C_0^\infty(\mathbb{R}^n)$.

It is worth noting that (4.16) is not true generally with BMO replaced by L^∞ . This is the reason for our usage of H^1 instead of L^1 . If (4.16) were true with L^∞ norm, we would obtain that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_2(x_2) f_3(x_3)}{|x_2|^{\alpha_{12}} |x_2 - x_3|^{\alpha_{23}} |x_3|^{\alpha_{13}}} dx_2 dx_3 \right| \leq C \|f_2\|_{p_3} \|f_3\|_{p_3} \quad (4.17)$$

for all $f_2, f_3 \in C_0^\infty(\mathbb{R}^n)$ by using Fatou's lemma. The integral in (4.17) is known as the Stein-Weiss potential. Under previous assumptions, (4.17) is valid if and only if $\frac{1}{p_2} + \frac{1}{p_3} \geq 1$. This can be proved by a similar argument in §3; see also [21]. But this additional assumption is not necessary for (4.16). This observation shows that BMO is an appropriate substitute for L^∞ in our main results.

Definition 4.1 *The space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions f satisfying*

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where the supremum is taken over all cubes with sides parallel to the axes and f_Q denotes the average of f over Q .

By taking modula relative to all constants, $BMO(\mathbb{R}^n)$ can be identified with a Banach space. This space was studied by John and Nirenberg in [15].

With above preliminaries, we turn to the proof of (4.16). By the above definition, we shall first show that $T(f_2, f_3)$ is locally integrable for arbitrary f_2, f_3 in C_0^∞ . This is guaranteed by (ii) in the system (IV). We shall present its simple proof in the next section. For any given cube Q with sides parallel to the axes, we divide $T(f_2, f_3)$ into three parts as follows:

$$\begin{aligned} T_1(f_2, f_3)(x_1) &= T(f_2 \chi_{*Q}, f_3) \\ T_2(f_2, f_3)(x_1) &= T(f_2, f_3 \chi_{*Q}) \\ T_3(f_2, f_3)(x_1) &= T(f_2 \chi_{*Q^c}, f_3 \chi_{*Q^c}). \end{aligned}$$

Here $*Q$ is the cube concentric with Q but expanded by two times and $*Q^c$ is the complement of $*Q$. We use χ_A to denote the characteristic function of A . Since the mean oscillation of $T(f_2, f_3)$ over Q is not greater than the summation of the mean oscillation of $T_i(f_2, f_3)$,

$$\frac{1}{|Q|} \int_Q |T(f_2, f_3)(x_1) - T(f_2, f_3)_Q| dx_1 \leq \sum_{i=1}^4 \frac{1}{|Q|} \int_Q |T_i(f_2, f_3)(x_1) - T_i(f_2, f_3)_Q| dx_1,$$

it suffices to show that each term in the summation is bounded by $C \|f_2\|_{p_2} \|f_3\|_{p_3}$ with the constant C independent of Q . For $i \in \{1, 2\}$, we shall see that the average of $|T_i(f_2, f_3)|$ over Q is not greater than a constant multiple of $\|f_2\|_{p_2} \|f_3\|_{p_3}$. We only prove this claim for T_1 since the argument is similar for T_2 . Observe that $\int_Q |T_1(f_2, f_3)| dx_1 \leq \Lambda(\chi_Q, |f_2| \chi_{*Q}, |f_3|)$. Choose

$\{\overline{p_i}\}_{i=1}^3$ such that (a) $1 < \overline{p_i} < \infty$; (b) $|f_2|\chi^*Q \in L^{\overline{p_2}}$ and $|f_3|\chi^*Q \in L^{\overline{p_3}}$; (c) $\{\overline{p_i}\}$ and $\{\alpha_{ij}\}$ satisfy (i), (ii) and (a) of (iii) in Theorem 1.1. Set

$$\overline{p_1} = 1 + \epsilon, \quad \overline{p_2} = \left(\frac{\epsilon}{1 + \epsilon} + \frac{1}{p_2} \right)^{-1} \quad \text{and} \quad \overline{p_3} = p_3.$$

If $\epsilon > 0$ is small enough, it is easily verified that $\{\overline{p_i}\}$ is just desired. The boundedness of Λ implies

$$\Lambda(\chi_Q, |f_2|\chi^*Q, |f_3|) \leq C|Q| \|f_2\|_{p_2} \|f_2\|_{p_3}.$$

Hence T_1 satisfies the claim. The same result is true for $T_2(f_2, f_3)$.

For the estimate of T_4 , first observe that if $x_2 \in {}^*Q^c$, $x_3 \in {}^*Q^c$, $x_1 \in Q$, we have

$$\begin{aligned} & \left| |x_1 - x_2|^{-\alpha_{12}} |x_1 - x_3|^{-\alpha_{13}} - (|\cdot - x_2|^{-\alpha_{12}} |\cdot - x_3|^{-\alpha_{13}})_Q \right| \\ & \leq C|Q|^{1/n} (|x_2 - c_Q|^{-1-\alpha_{12}} |x_3 - c_Q|^{-\alpha_{13}} + |x_2 - c_Q|^{-\alpha_{12}} |x_3 - c_Q|^{-1-\alpha_{13}}). \end{aligned} \quad (4.18)$$

If either α_{12} or α_{13} were equal to zero, the right side of the above inequality would reduce to one term. Indeed, if $\alpha_{13} = 0$, then $\alpha_{23}/n + 1/p_3 > 1$ and the average of $|T_2(f_2, f_3)(x_1) - T_2(f_2, f_3)_Q|$ over Q is bounded by a constant multiple of

$$\begin{aligned} |Q|^{1/n} \int_{{}^*Q^c} \int_{{}^*Q^c} \frac{|f_2(x_2)f_3(x_3)|}{|x_2 - c_Q|^{1+\alpha_{12}} |x_2 - x_3|^{\alpha_{23}}} dx_2 dx_3 &= |Q|^{1/n} \int_{{}^*Q^c} |\tilde{f}_2(x_2) I_{n-\alpha_{23}}(|f_3|)(x_2)| dx_2 \\ &\leq |Q|^{1/n} \|\tilde{f}_2\|_{L^{q'_3}({}^*Q^c)} \|I_{n-\alpha_{23}}(|f_3|)\|_{q_3} \\ &\leq C \|f_2\|_{p_2} \|f_3\|_{p_3} \end{aligned}$$

with $1 + 1/q_3 = \alpha_{23}/n + 1/p_3$ and $\tilde{f}_2(x) = f_2(x)|x - c_Q|^{-\alpha_{12}-1}$, where we have used the Hölder inequality $\|\tilde{f}_2\|_{L^{q'_3}({}^*Q^c)} \leq C|Q|^{-1/n} \|f_2\|_{p_2}$ and the validity of above inequalities is justified by assumptions in the system (IV). The treatment is similar in the case $\alpha_{12} = 0$.

Now assume $\alpha_{12} > 0$ and $\alpha_{13} > 0$. By insertion of each term of (4.18) into the fractional integral, the argument is similar. For the first term, we have

$$\begin{aligned} & \int_{{}^*Q^c} \int_{{}^*Q^c} \frac{|f_2(x_2)f_3(x_3)|}{|x_2 - c_Q|^{1+\alpha_{12}} |x_3 - c_Q|^{\alpha_{13}} |x_2 - x_3|^{\alpha_{23}}} dx_2 dx_3 \\ &= \int_{{}^*Q^c} \int_{{}^*Q^c} \frac{|\tilde{f}_2(x_2)\tilde{f}_3(x_3)|}{|x_2 - x_3|^{\alpha_{23}}} dx_2 dx_3 \\ &\leq C \|\tilde{f}_2\|_{q_2} \|\tilde{f}_3\|_{q_3}, \end{aligned}$$

where

$$f_2(x_2) = |x_2 - c_Q|^{-1-\alpha_{12}} f_2(x_2), \quad f_3(x_3) = |x_3 - c_Q|^{-\alpha_{13}} f_3(x_3),$$

q_2 and q_3 satisfy $1 < q_2, q_3 < n/(n - \alpha_{23})$ and $1/q_2 + 1/q_3 + \alpha_{23}/n = 2$. Applying the Hölder inequality, we get

$$\|\tilde{f}_2\|_{q_2} \leq C|Q|^{1/s_2 - (1+\alpha_{12})/n} \|f_2\|_{p_2} \quad \text{and} \quad \|\tilde{f}_3\|_{q_3} \leq C|Q|^{1/s_3 - \alpha_{13}/n} \|f_3\|_{p_3}$$

for $1/p_2 + 1/s_2 = 1/q_2$ and $1/p_3 + 1/s_3 = 1/q_3$, where $s_2 > n/(1 + \alpha_{12})$ and $s_3 > n/\alpha_{13}$ are to be determined. Since $\alpha_{13} + n/p_3 < n$ and $\alpha_{12} + n/p_2 < n$, we can choose $0 < \varepsilon < \alpha_{13}$ sufficiently small and $s_3 = n/(\alpha_{13} - \varepsilon)$ such that

$$1 - \frac{\alpha_{23}}{n} < \frac{1}{p_3} + \frac{\alpha_{13} - \varepsilon}{n} = \frac{1}{q_3} < 1, \quad 1 - \frac{\alpha_{23}}{n} < \frac{1}{p_2} + \frac{\alpha_{12} + \varepsilon}{n} = \frac{1}{q_2} < 1.$$

As a result, it follows that $s_2 = n/(\alpha_{12} + \epsilon)$ is greater than $n/(1 + \alpha_{12})$ with sufficiently small ϵ . And $s_3 = n/(\alpha_{13} - \epsilon) > n/\alpha_{13}$. Thus the mean oscillation of T_4 over Q is bounded by a constant multiple of $\|f_2\|_{p_2}\|f_3\|_{p_3}$.

Combining above estimates, we complete the proof.

5 Locally integrable conditions and L^∞ estimates

In this section, we shall establish the necessary and sufficient conditions under which the multilinear operator T in the introduction is bounded from $(C_0^\infty(\mathbb{R}^n))^k$ into $\mathcal{D}'(\mathbb{R}^n)$. This is equivalent to saying that the Selberg integral of the correlation kernel $\prod |x_i - x_j|^{-\alpha_{ij}}$ is finite on any bounded region in $\mathbb{R}^{n(k+1)}$. The argument in this section turns out to be very useful throughout subsequent sections.

Theorem 5.1 *Assume $\alpha_{ij} \geq 0$ for $1 \leq i < j \leq k+1$ satisfy the integrable conditions*

$$\sum_J \alpha_{ij} < (|J| - 1)n \quad (5.19)$$

for any subset J of $\{1, 2, \dots, k+1\}$ with $|J| \geq 2$. Then we have

$$\int_{(B_1(0))^{k+1}} \prod_{\{1, 2, \dots, k+1\}} |x_i - x_j|^{-\alpha_{ij}} dx_1 dx_2 \cdots dx_{k+1} < \infty, \quad (5.20)$$

where $B_1(0) \subset \mathbb{R}^n$ is the unit ball centered at the origin.

Proof. In the case $k = 1$, it is clear that the above integral converges absolutely. For $k \geq 2$, we begin with the simplest case $k = 2$ and then make induction for general k . For $k = 2$, it is convenient to divide the proof into three cases.

Case 1. $\alpha_{13} + \alpha_{23} < n$

It is clear that

$$\begin{aligned} & \int_{(B_1(0))^3} \prod_S |x_i - x_j|^{-\alpha_{ij}} dx_1 dx_2 dx_3 \\ & \leq C \int_{(B_1(0))^2} |x_1 - x_2|^{-\alpha_{12}} dx_1 dx_2 \end{aligned}$$

which is finite by the assumption $\alpha_{12} < n$.

Case 2. $\alpha_{13} + \alpha_{23} = n$

Using Lemma 2.4, we obtain

$$\begin{aligned} & \int_{(B_1(0))^3} \prod_S |x_i - x_j|^{-\alpha_{ij}} dx_1 dx_2 dx_3 \\ & \leq C \int_{(B_1(0))^2} |x_1 - x_2|^{-\alpha_{12}} \log(4/|x_1 - x_2|) dx_1 dx_2 < \infty. \end{aligned}$$

Case 3. $\alpha_{13} + \alpha_{23} > n$

Observe that

$$\int_{B_1(0)} |x_1 - x_3|^{-\alpha_{13}} |x_2 - x_3|^{-\alpha_{23}} dx_3 \leq C |x_1 - x_2|^{n - \alpha_{13} - \alpha_{23}}$$

which implies the integral in (5.20) is finite by the assumption $\alpha_{12} + \alpha_{13} + \alpha_{23} < 2n$.

We now consider the general case $k \geq 4$. Assume the statement holds for $k - 1$ under the assumption (5.19) with $k \geq 3$. We shall prove the statement is also valid for k . Let

$$\Theta = \{i : 1 \leq i \leq k, \quad \alpha_{i,k+1} > 0\}.$$

By simple calculations, it is easy to verify our claim in the case $\sum_{\Theta} \alpha_{i,k+1} \leq n$. Indeed, if $\sum_{\Theta} \alpha_{i,k+1}$ is less than n , we take integration with respect to x_{k+1} and then the matter reduces to the case $k - 1$. If $\sum_{\Theta} \alpha_{i,k+1} = n$, it follows from Lemma 2.4 that

$$\int_{B_1(0)} \prod_{\Theta} |x_i - x_{k+1}|^{-\alpha_{i,k+1}} dx_{k+1} \leq C \left(\sum_{\Theta} |x_i - x_j| \right)^{-\varepsilon}$$

where $\varepsilon > 0$ is a small number to be determined. Choose $i_0, j_0 \in \Theta$ with $i_0 < j_0$. Let

$$\overline{\alpha_{ij}} = \alpha_{ij} + \delta_i^{i_0} \delta_j^{j_0} \epsilon, \quad 1 \leq i < j \leq k,$$

where δ_s^t is the Kronecker symbol. In other words, $\delta_s^t = 1$ if $s = t$ and $\delta_s^t = 0$ otherwise. If ϵ is sufficiently small, then $\{\overline{\alpha_{ij}}\}$ still satisfies the integrable conditions. Therefore the $k + 1$ -multiple integral (5.20) is less than a constant multiple of a k -multiple integral of the same form with $\{\alpha_{ij}\}$ replaced by $\{\overline{\alpha_{ij}}\}$. The desired result follows by induction.

The crux of the proof is the result for $\sum_{\Theta} \alpha_{i,k+1} > n$. The argument depends on the number of elements in Θ . The simplest case is $|\Theta| = 2$ in which the argument is direct. For $3 \leq |\Theta| \leq k$, we shall reduce the statement to the case $|\Theta| = 2$ by using a useful procedure. Indeed, if $|\Theta| = 2$, we may assume $\Theta = \{1, 2\}$ by the symmetry of parameters. Then by a similar estimate in Case 3 for $k = 2$, put

$$\overline{\alpha_{ij}} = \alpha_{ij} + \delta_i^1 \delta_j^2 \left(\sum_{\Theta} \alpha_{i,k+1} - n \right), \quad 1 \leq i < j \leq k. \quad (5.21)$$

It is easy to verify that the integrable conditions (5.19) are still true for $\{\overline{\alpha_{ij}}\}$. Indeed, it suffices to show that (5.19) holds for those J containing both 1 and 2 since we obviously have

$$\sum_J \alpha_{ij} = \sum_J \overline{\alpha_{ij}}$$

for those subsets J of $S = \{1, 2, \dots, k\}$ not containing $\{1, 2\}$. For $J \subset S$ containing Θ , by the definition of $\{\overline{\alpha_{ij}}\}$ it follows that

$$\sum_J \overline{\alpha_{ij}} = \sum_{J \cup \{k+1\}} \alpha_{ij} - n$$

which is less than $(|J| - 1)n$ by the assumption (5.19). Hence the integral in (5.20) converges by induction in the case $|\Theta| = 2$.

If $|\Theta| = m$ with $3 \leq m \leq k$, our idea is to show that the $k + 1$ -multiple integral is dominated by sums of two kinds of similar integrals by adding some powers into $\{\alpha_{ij} : 1 \leq i < j \leq k\}$ appropriately. The first type of these integrals has a k -point correlation integrand. The other type is the same as the integral in (5.20) but with $|\Theta| = m - 1$. Without loss of generality, we may assume $\Theta = \{1, 2, \dots, m\}$. By the assumption $\sum_{\Theta} \alpha_{i,k+1} > n$, we use Lemma 2.3 to obtain

$$\int_{\mathbb{R}^n} \prod_{\Theta} |x_i - x_j|^{-\alpha_{i,k+1}} dx_{k+1} \leq C \sum_{\Theta} L_i, \quad (5.22)$$

where L_i are given as in Lemma 2.3.

Replacing the integral relative to x_{k+1} by each L_i , we shall prove that the $k+1$ multiple integral is dominated by integrals of the above two types.

If $\sum_{i=2}^m \alpha_{i,k+1} < n$, we shall prove that

$$\begin{aligned} & \int_{(B_1(0))^k} \left(\prod_S |x_i - x_j|^{-\alpha_{ij}} \right) \left(\sum_{\Theta} |x_i - x_j| \right)^{n - \sum_{\Theta} \alpha_{i,k+1}} dx_1 \cdots dx_k \\ & \leq C \int_{(B_1(0))^k} \prod_S |x_i - x_j|^{-\alpha_{ij} - \delta_{ij}} dx_1 \cdots dx_k, \end{aligned}$$

where $\{\overline{\alpha}_{ij} = \alpha_{ij} + \delta_{ij}\}$ satisfies the integrable conditions (5.19) for $J \subset S$. Here $\sum_{\Theta} \delta_{ij} = \sum_{\Theta} \alpha_{i,k+1} - n$ for $\delta_{ij} \geq 0$ and $\delta_{ij} = 0$ if either i or j not in Θ . We turn our attention to the existence of such a solution $\{\delta_{ij}\}$ now. In other words, our objective is to solve the following system of linear inequalities:

$$(V.1) \begin{cases} (i) & \delta_{ij} \geq 0, \quad 1 \leq i < j \leq m; \\ (ii) & \sum_{\Theta} \delta_{ij} = \sum_{\Theta} \alpha_{i,k+1} - n; \\ (iii) & \sum_{J \cap \Theta} \delta_{ij} < (|J| - 1)n - \sum_J \alpha_{ij} \quad \text{for } J \in \mathcal{F}_m, \end{cases}$$

where the class \mathcal{F}_m consists of all subsets J of $\{1, \dots, k\}$ satisfying $|J \cap \Theta| \geq 2$. Note that we have assumed $\Theta = \{1, 2, \dots, m\}$. Here we use the notation \mathcal{F}_m instead of \mathcal{F}_{Θ} for simplicity.

In the following argument, Lemma 2.5 is the main tool to show the existence of solutions to the system (V.1). For arbitrary nonnegative numbers λ_{ij} , θ_1 , θ_2 and μ_J with at least one $\mu_J > 0$ for some J in the class \mathcal{F}_m satisfying

$$\lambda_{ij} + (\theta_1 - \theta_2) - \sum_{J \ni i,j} \mu_J = 0 \quad (5.23)$$

for $1 \leq i < j \leq m$, we must show that

$$(\theta_1 - \theta_2) \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{F}_m} \mu_J \left((|J| - 1)n - \sum_J \alpha_{ij} \right) < 0. \quad (5.24)$$

It suffices to prove this inequality when $\theta_1 - \theta_2 > 0$ since there exists one $\mu_J > 0$ and $\sum_J \alpha_{ij}$ is less than $(|J| - 1)n$. Now assume $\theta_1 - \theta_2 > 0$. By dilation, put $\theta_1 - \theta_2 = 1$. Then μ_J and λ_{ij} satisfy

$$\sum_{J \ni i,j} \mu_J = 1 + \lambda_{ij} \quad (5.25)$$

for $1 \leq i < j \leq m$. This reduction to $\theta_1 - \theta_2 = 1$ does not simplify the proof of the inequality (5.24). To prove this inequality, a basic idea is to determine the maximum of the objective function on the left side of the inequality and then show this maximum is negative. Though the maximum cannot be attained generally, parameters μ_J and λ_{ij} have simple characters when the value of the objective function is close to its maximum sufficiently. More precisely, for any set of parameters $\{\mu_J(0), \lambda_{ij}(0)\}$ in the domain of μ_J and λ_{ij} , we will get a sequence of sets $\{\mu_J(N), \lambda_{ij}(N)\}$ such that the object function does not decrease when $\{\mu_J(0), \lambda_{ij}(0)\}$ replaced by $\{\mu_J(N), \lambda_{ij}(N)\}$. By taking $N \rightarrow \infty$, the sign of the object function will be easily verified.

Now we turn to construct such a process. If $\{\mu_J(N-1)\}$ and $\{\lambda_{ij}(N-1)\}$ are known, then we choose J_1 and J_2 such that $\mu_{J_1}(N-1)\mu_{J_2}(N-1) > 0$ and $J_1 \cap J_2 \neq \emptyset$. Then we set $\mu_J(N)$ as follows.

Case I: $J_1, J_2 \in \mathcal{F}_m$ and $J_1 \cap J_2 \in \mathcal{F}_m$ with $\mu_{J_1}(N-1)\mu_{J_2}(N-1) > 0$.

$$\begin{cases} \mu_{J_1}(N) = \mu_{J_1}(N-1) - \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \\ \mu_{J_2}(N) = \mu_{J_2}(N-1) - \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \\ \mu_{J_1 \cap J_2}(N) = \mu_{J_1 \cap J_2}(N-1) + \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \\ \mu_{J_1 \cup J_2}(N) = \mu_{J_1 \cup J_2}(N-1) + \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \\ \mu_J(N) = \mu_J(N-1), \quad J \notin \{J_1, J_2, J_1 \cap J_2, J_1 \cup J_2\}. \end{cases} \quad (5.26)$$

If $J_1 \subset J_2$ and $J_2 \subset J_1$, then this process does not change the values of μ_J .

Case II: $J_1, J_2 \in \mathcal{F}_m$, $J_1 \cap J_2 \notin \mathcal{F}_m$ and $|J_1 \cap J_2| \geq 1$ with $\mu_{J_1}(N-1)\mu_{J_2}(N-1) > 0$.

$$\begin{cases} \mu_{J_1}(N) = \mu_{J_1}(N-1) - \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \\ \mu_{J_2}(N) = \mu_{J_2}(N-1) - \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \\ \mu_{J_1 \cup J_2}(N) = \mu_{J_1 \cup J_2}(N-1) + \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \\ \mu_J(N) = \mu_J(N-1), \quad J \notin \{J_1, J_2, J_1 \cup J_2\}. \end{cases} \quad (5.27)$$

Here $A \wedge B = \min\{A, B\}$. We also define $\lambda_{ij}(N)$ by (5.25) with $\{\mu_J, \lambda_{ij}\}$ replaced by $\{\mu_J(N), \lambda_{ij}(N)\}$. The motivation for the construction of such a process is the following inequality,

$$\begin{aligned} & \left((|J_1 \cap J_2| - 1)n - \sum_{J_1 \cap J_2} \alpha_{ij} \right) + \left((|J_1 \cup J_2| - 1)n - \sum_{J_1 \cup J_2} \alpha_{ij} \right) \\ & \leq \sum_{s=1}^2 \left((|J_s| - 1)n - \sum_{J_s} \alpha_{ij} \right) \end{aligned} \quad (5.28)$$

for all J_1 and J_2 in the class \mathcal{F}_m . Here we use the summation convention $\sum_J \alpha_{ij} = 0$ if $|J| \leq 1$.

It is helpful to make some observations of the above process. First we claim that $\lambda_{ij}(N)$ increases as N . Assume that $\mu_{J_1}(N-1)$ and $\mu_{J_2}(N-1)$ have been changed in the N -th step. For each pair i and j , there are several possible cases. If either $i \notin J_1 \cup J_2$ or $j \notin J_1 \cup J_2$, then $\lambda_{ij}(N) = \lambda_{ij}(N-1)$. Now we treat the main case $i, j \in J_1 \cup J_2$ and divide it into three subcases. If $i, j \in J_1 \cap J_2$, then it is easy to see that $\lambda_{ij}(N) = \lambda_{ij}(N-1)$. If $i \notin J_1 \cap J_2$ or $j \notin J_1 \cap J_2$ but either $i, j \in J_1$ or $i, j \in J_2$, then λ_{ij} remains unchanged in the N step. The remaining subcase is that $i, j \notin J_1$ and $i, j \notin J_2$ in which λ_{ij} increases in the N step. Thus we have established our claim.

The key observation is that the object function also increases as N . Equivalently, we have

$$- \sum_{J \in \mathcal{F}_m} \mu_J(N-1) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \leq - \sum_{J \in \mathcal{F}_m} \mu_J(N) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \quad (5.29)$$

for any $N \geq 1$. Indeed, this observation is an immediate consequence of the inequality (5.28) and its simple variant

$$(|J_1 \cup J_2| - 1)n - \sum_{J_1 \cup J_2} \alpha_{ij} \leq \sum_{s=1}^2 \left((|J_s| - 1)n - \sum_{J_s} \alpha_{ij} \right)$$

with the additional assumption $|J_1 \cap J_2| \geq 1$. This explains why the process (5.27) only applies to those J_1 and J_2 satisfying $|J_1 \cap J_2| \geq 1$.

Now we shall introduce some subclasses of \mathcal{F}_m . Let \mathcal{A}_m , \mathcal{B}_m and \mathcal{C}_m be defined by

$$\mathcal{A}_m = \{J \in \mathcal{F}_m : \mu_J > 0\}, \quad \mathcal{B}_m = \{J \in \mathcal{A}_m : \Theta \not\subseteq J\}, \quad \mathcal{C}_m = \{J \in \mathcal{A}_m : \Theta \subseteq J\}. \quad (5.30)$$

For clarity, we also use $\mathcal{A}_m(N)$, $\mathcal{B}_m(N)$ and $\mathcal{C}_m(N)$ defined similarly as above to keep track of the above process. It is clear that $\mathcal{B}_m \cap \mathcal{C}_m = \emptyset$ and $\mathcal{A}_m = \mathcal{B}_m \cup \mathcal{C}_m$.

Definition 5.1 *If $\{\mu_J : J \in \mathcal{F}_m\}$ is invariant under any possible process described as in (5.26) and (5.27), then we say that the set $\{\mu_J : J \in \mathcal{F}_m\}$ is stable.*

Let $F(\{\mu_J\})$ be a function of $\mu_J \geq 0$ for all $J \in \mathcal{F}_m$. Assume $\{\mu_J(0) : J \in \mathcal{F}_m\}$ is a set of nonnegative numbers. We say that F is stable with respect to $\{\mu_J(0)\}$ if for all $N \geq 1$ $F(\{\mu_J(0)\}) = F(\{\mu_J(N)\})$, where $\{\mu_J(N)\}$ is obtained by an arbitrary process described as in (5.26) and (5.27) within N steps.

By this definition, $\{\mu_J : J \in \mathcal{F}_m\}$ is stable if and only if for all $J_1, J_2 \in \mathcal{A}_m$ one of the three relations holds, (i) $J_1 \cap J_2 = \emptyset$, (ii) $J_1 \subset J_2$ and (iii) $J_2 \subset J_1$. Further observation also shows $\sum_{J \in \mathcal{C}_m} \mu_J = 1 + \min_{\Theta} \lambda_{ij}$ when μ_J are stable. This observation will be proved later. For convenience, we introduce the notation Ω_m to denote

$$\Omega_m = \sum_{J \in \mathcal{C}_m} \mu_J \quad (5.31)$$

And $\Omega_m(N)$ is defined as above with \mathcal{C}_m and μ_J replaced by $\mathcal{C}_m(N)$ and $\mu_J(N)$ respectively.

We do not know whether any $\{\mu_J(0) : J \in \mathcal{F}_m\}$ and $\{\lambda_{ij}(0)\}$ satisfying (5.25) can reach a stable state by a process consisting of finite steps. However, by a passage to the limit, we will arrive at a special state, not stable generally, which is enough for our purpose. Let $\{\mu_J^*(N) : J \in \mathcal{F}_m\}$ be obtained by a process of N steps described as in (5.26) and (5.27) such that the supremum

$$\Omega_m^*(N) = \sup \Omega_m(N)$$

is attained. Here the supremum is taken over all possible processes consisting of N steps. It is possible that these processes are not unique. In other words, there are more than two ways to obtain $\{\mu_J^*(N) : J \in \mathcal{F}_m\}$. We can take one of these processes by which $\Omega_m^*(N)$ is obtained. It should be mentioned that $\{\mu_J^*(N)\}$ is not obtained by a continuous procedure with respect to N . Therefore in general we cannot obtain $\{\mu_J^*(N)\}$ from $\{\mu_J^*(N-1)\}$ by one step. On the one hand, it is clear that $\Omega_m^*(N)$ increases as N . On the other hand, we also have

$$\sum_{J \ni i} \mu_J^*(N) \leq \sum_{J \ni i} \mu_J(0) \leq \max_{1 \leq i \leq k} \left(\sum_{J \ni i} \mu_J(0) \right), \quad (5.32)$$

for any $i \in S = \{1, 2, \dots, k\}$. Suppose at the k -th step with $k \leq N-1$ the process is applied to J_1 and J_2 in \mathcal{F}_m . Then we see that $\sum_{J \ni i} \mu_J(k) = \sum_{J \ni i} \mu_J(k+1)$ unless $i \in J_1 \cap J_2$ and $J_1 \cap J_2 \notin \mathcal{F}_m$. In the latter case, we have $\sum_{J \ni i} \mu_J(k) > \sum_{J \ni i} \mu_J(k+1)$. There is another similar observation as (5.32). In the system (V.1), δ_{ij} is assumed to be zero if $i \notin \Theta$ or $j \notin \Theta$. If $\{\mu_J(0)\}$ and $\{\lambda_{ij}(0)\}$ satisfy the equations (5.25), we may assume that $\mu_J(0) = 0$ for nonempty $J \subset S$ but $J \notin \mathcal{F}_m$. Then we see that $\mu_J(N) = 0$ if $J \notin \mathcal{F}_m$ for any N . Thus we obtain

$$\sum_{J \subset S} \mu_J(N) \leq \sum_{J \subset S} \mu_J(0), \quad (5.33)$$

where $\mu_J(N)$ are obtained by any possible process with N steps. Both (5.32) and (5.33) imply that $\{\Omega_m^*(N)\}$ has a uniform upper bound. Put

$$\Omega_m^*(\infty) = \lim_{N \rightarrow \infty} \Omega_m^*(N). \quad (5.34)$$

This limit is well defined since $\{\Omega_m^*(N)\}$ is a bounded and increasing sequence. It is possible that $\Omega_m^*(\infty)$ is obtained by a process of finite steps, i.e., $\Omega_m^*(\infty) = \Omega_m^*(N)$ for $N \geq N_0$ with a large N_0 . In this situation, $\Omega_m^*(N_0)$ is a stable state in the following. Indeed, we shall give a necessary and sufficient condition under which Ω_m is stable with respect to $\{\mu_J\}$. It will be convenient to introduce a concept related to a sequence of sets.

Definition 5.2 *If J_1, J_2, \dots, J_a with $a \geq 2$ is a sequence of sets with the property that the intersection $(\bigcup_{t=1}^i J_t) \cap J_{i+1}$ is nonempty for $1 \leq i \leq a-1$, then we call $\{J_i\}_{i=1}^a$ a continuous chain.*

Lemma 5.2 *Assume $\{\mu_J : J \in \mathcal{F}_m\}$ is a set of nonnegative numbers and Ω_m is defined by (5.31). Then Ω_m is stable with respect to $\{\mu_J\}$ if and only if $\Theta \not\subseteq \bigcup_{i=1}^a J_i$ for any continuous chain $\{J_i\}_{i=1}^a$ in \mathcal{B}_m .*

Proof. We first show the necessity part. Assume the converse. Then there exists a continuous chain $\{J_i\}_{i=1}^a$ in $\mathcal{B}_m(0)$ such that $\Theta \subset \bigcup_{i=1}^a J_i$. We shall see that there is a process such that $\Omega_m(a-1) > \Omega_m(0)$. First applying the recursion described as in (5.26) and (5.27) to J_1 and J_2 , then we obtain $\{\mu_J(1)\}$. In the second step, we repeat the process with respect to $J_1 \cup J_2$ and J_3 . Likewise, in the i -th step we apply the recursion to $\bigcup_{1 \leq t \leq i} J_t$ and J_{i+1} . After $a-1$ steps, we shall see that

$$\Omega_m(a-1) \geq \Omega_m(0) + \min_{1 \leq i \leq a} \mu_{J_i}(0) \geq \Omega_m(0) + \min_{J \in \mathcal{A}_m} \mu_J(0) \quad (5.35)$$

which contradicts the assumption that $\Omega_m(0)$ is stable.

The proof of the sufficiency is intricate. We first establish a useful property of \mathcal{B}_m under the assumption that for all continuous chains $\{J_i\}_{i=1}^a$ in \mathcal{B}_m , the union $\bigcup_{i=1}^a J_i$ does not contain Θ as a subset.

Proposition 5.3 *Suppose that Θ is not a subset of the union $\bigcup_{i=1}^a J_i$ for all continuous chains $\{J_i\}_{i=1}^a$ in \mathcal{B}_m . Then there exists a nonempty and proper subset Θ_1 of Θ such that for all J_1 and J_2 in \mathcal{B}_m with $J_1 \cap J_2 \neq \emptyset$, one of the following relations is true:*

$$(J_1 \cup J_2) \cap \Theta \subset \Theta_1 \quad \text{or} \quad (J_1 \cup J_2) \cap \Theta \subset \Theta - \Theta_1. \quad (5.36)$$

Generally, one of the above two relations also holds for all continuous chains $\{J_i\}_{i=1}^a$ in \mathcal{B}_m . More precisely, we have

$$\text{either } \left(\bigcup_{i=1}^a J_i \right) \cap \Theta \subset \Theta_1 \quad \text{or} \quad \left(\bigcup_{i=1}^a J_i \right) \cap \Theta \subset \Theta - \Theta_1.$$

Now we turn to prove the proposition. Let η be given by

$$\eta = \sup \left| \left(\bigcup_{i=1}^a J_i \right) \cap \Theta \right| \quad (5.37)$$

where the supremum is taken over all continuous chains $\{J_i\}$ in \mathcal{B}_m . It is easy to see that η can be obtained for some continuous chain $\{I_i\}_{i=1}^a$ in \mathcal{B}_m . Let $\Theta_1 = (\bigcup_{i=1}^a I_i) \cap \Theta$. Then Θ_1 is a proper subset of Θ and for all J in \mathcal{B}_m either $J \cap \Theta \subset \Theta_1$ or $J \cap (\bigcup_{i=1}^a I_i) = \emptyset$ is true. Otherwise, if there were some $I_{a+1} \in \mathcal{B}_m$ such that $I_{a+1} \cap \Theta \not\subset \Theta_1$ and $I_{a+1} \cap (\bigcup_{i=1}^a I_i) \neq \emptyset$, it would follow that $\{I_i\}_{i=1}^{a+1}$ is a continuous chain in \mathcal{B}_m and satisfies $\left| \left(\bigcup_{i=1}^{a+1} I_i \right) \cap \Theta \right| > \eta$. This contradicts the definition of η .

Define $\mathcal{B}_m^{(1)}$ and $\mathcal{B}_m^{(2)}$ by

$$\mathcal{B}_m^{(1)} = \left\{ J \in \mathcal{B}_m : J \cap \left(\bigcup_{i=1}^a I_i \right) = \emptyset \right\}, \quad \mathcal{B}_m^{(2)} = \mathcal{B}_m - \mathcal{B}_m^{(1)}.$$

Then $J \cap \Theta \subset \Theta_1$ for all $J \in \mathcal{B}_m^{(2)}$. Moreover, we also have that $J_1 \cap J_2 = \emptyset$ if $J_1 \in \mathcal{B}_m^{(1)}$ and $J_2 \in \mathcal{B}_m^{(2)}$. Actually, if there were $J_1 \in \mathcal{B}_m^{(1)}$ and $J_2 \in \mathcal{B}_m^{(2)}$ with $J_1 \cap J_2 \neq \emptyset$, it would follow that $\{I_i\}_{i=1}^{a+2}$ is a continuous chain in \mathcal{B}_m and $\left| \left(\bigcup_{i=1}^{a+2} I_i \right) \cap \Theta \right| > \eta$ where $I_{a+1} = J_2$ and $I_{a+2} = J_1$. This contradicts our choice of η . Therefore one of two relations in (5.36) is true if $J_1 \cap J_2 \neq \emptyset$. It remains to prove that the same conclusion holds for any continuous chain in \mathcal{B}_m . Without loss of generality, we may assume $a = 3$. Since the intersection of J_1 and J_2 is nonempty, we see that either $J_1, J_2 \in \mathcal{B}_m^{(1)}$ or $J_1, J_2 \in \mathcal{B}_m^{(2)}$ is true. By the assumption that J_1, J_2, J_3 is a continuous chain, we have either $J_1 \cap J_3 \neq \emptyset$ or $J_2 \cap J_3 \neq \emptyset$. The desired conclusion holds by combining above observations. We conclude the proof of the proposition.

Now we shall invoke Proposition 5.3 to give a complete proof of the sufficiency of Lemma 5.2. For initially given data $\{\mu_J(0)\}$ and $\{\lambda_{ij}(0)\}$ of nonnegative numbers satisfying (5.25), we assume that there is a nonempty $\Theta_1 \subsetneq \Theta$ such that for all $J_1, J_2 \in \mathcal{B}_m(0)$ satisfying $J_1 \cap J_2 \neq \emptyset$, one of two relations in (5.36) is true. We shall prove that $\Omega_m(0)$ is stable. In other words, $\Omega_m(0) = \Omega_m(N)$ for a process with N steps. Here $\{\mu_J(N)\}$ is obtained from $\{\mu_J(0)\}$ by any process consisting of N steps described as (5.26) and (5.27). At the k -th step in which $\{\mu_J(k)\}$ is obtained, assume that we apply the recursion to J_1 and J_2 in $\mathcal{A}_m(k-1)$. Then $J_1 \cap J_2 \neq \emptyset$. We divide the k -th step into four cases:

- (1) J_1 and J_2 are in $\mathcal{C}_m(k-1)$; (2) $J_1 \in \mathcal{B}_m(k-1)$ and $J_2 \in \mathcal{C}_m(k-1)$;
- (3) $J_1 \in \mathcal{C}_m(k-1)$ and $J_2 \in \mathcal{B}_m(k-1)$; (4) J_1 and J_2 are in $\mathcal{B}_m(k-1)$.

We claim that $\Omega_m(k-1) = \Omega_m(k)$. For J_1 and J_2 in the cases (1), (2) and (3), this statement is easily verified. In the case (4), we need an additional property of $\mathcal{B}_m(k)$. This property states that the result in Proposition 5.3 is still true for $\mathcal{B}_m(k)$ with the same Θ_1 . Since $\mathcal{B}_m(0)$ is assumed to have the property stated in Proposition 5.3, it suffices to show that $\mathcal{B}_m(1)$ shares this property. Assume in the first step, the recursion applies to J_1 and J_2 in \mathcal{F}_m . Observe that $J_1 \cap J_2$ and $J_1 \cup J_2$ are the only two subsets which are possibly contained in $\mathcal{B}_m(1)$ but not in $\mathcal{B}_m(0)$. It follows from the above proposition that both intersections $(J_1 \cap J_2) \cap \Theta$ and $(J_1 \cup J_2) \cap \Theta$ are subsets of Θ_1 or $\Theta - \Theta_1$. Hence the assumption in the proposition is also valid for $\mathcal{B}_m(1)$. By induction, it follows that the same result holds for all $\mathcal{B}_m(k)$. Thus for the case (4) mentioned above, we still have $\Omega(k) = \Omega(k-1)$. This implies that $\Omega(0)$ is stable. The proof of Lemma 5.2 is complete. \square

We now turn our attention to prove that $\Omega_m^*(\infty)$ is stable. Recall that both (5.32) and (5.33) imply the uniform boundedness of $\mu_J^*(N)$ and $\lambda_{ij}^*(N)$. Thus by passing $\{N\}$ to a subsequence, denoted by $\{N_t\}$, such that $\mu_J^*(N_t)$ and $\lambda_{ij}^*(N_t)$ converge to $\mu_J^*(\infty)$ and $\lambda_{ij}^*(\infty)$ respectively. It

is easy to see that the limits $\mu_J^*(\infty)$ and $\lambda_{ij}^*(\infty)$ still satisfy (5.25). By the definition of $\Omega_m^*(\infty)$, we have

$$\Omega_m^*(\infty) = \sum_{J \in \mathcal{C}_m(\infty)} \mu_J^*(\infty)$$

with $\mathcal{C}_m(\infty)$ consisting of all those $J \in \mathcal{F}_m$ satisfying $\Theta \subset J$ and $\mu_J^*(\infty) > 0$. Let $\mathcal{A}_m(\infty)$ be the class of all $J \in \mathcal{F}_m$ satisfying $\mu_J^*(\infty) > 0$. Similarly, $\mathcal{B}_m(\infty)$ can be defined as (5.30). With these notations, we claim that $\Omega_m(\infty)$ is stable with respect to the class $\mu_J(\infty)$. This means that any application of above processes to $\mathcal{A}_m(\infty)$ does not change the value of $\Omega_m(\infty)$. Otherwise, by Lemma 5.2, there exists a continuous chain $\{J_i\}_{i=1}^a$ in $\mathcal{B}_m(\infty)$ satisfying $\Theta \subset \bigcup_{i=1}^a J_i$. First observe that $\{\mu_J^*(N_t)\}$ has a uniform positive lower bound for each $J \in \mathcal{A}_m(\infty)$ when $t \geq t_0$ with some sufficiently large t_0 . By (5.35) and related remarks, it follows that

$$\Omega_m^*(N_t + a - 1) \geq \Omega_m^*(N_t) + \min_{J \in \mathcal{A}(\infty)} \mu_J^*(N_t) \geq \Omega_m^*(N_t) + \frac{1}{2} \min_{J \in \mathcal{A}(\infty)} \mu_J^*(\infty) \quad (5.38)$$

for all sufficiently large N_t . This contradicts the fact that $\Omega_m^*(N)$ converges to $\Omega_m^*(\infty) = \Omega_m(\infty)$.

Another simple but key observation is

$$\Omega_m^*(\infty) = 1 + \min_{\Theta} \lambda_{ij}^*(\infty) \geq 1 + \min_{\Theta} \lambda_{ij}(0). \quad (5.39)$$

Actually, we can choose a continuous chain $\{I_i\}_{i=1}^a$ from $\mathcal{B}_m(\infty)$ such that $|\bigcup_{i=1}^a I_i \cap \Theta|$ is the largest over all continuous chains in $\mathcal{B}_m(\infty)$. Choose $i_0 \in (\bigcup_{i=1}^a I_i) \cap \Theta$ and $j_0 \in \Theta$ but $j_0 \notin (\bigcup_{i=1}^a I_i) \cap \Theta$. Then $\mu_J^*(\infty) = 0$ for any $J \in \mathcal{B}_m(\infty)$ satisfying $i_0, j_0 \in J$. Then (5.25) becomes $\sum_{J \supset \Theta} \mu_J^*(\infty) = 1 + \lambda_{ij}^*(\infty)$ with $(i, j) = (i_0, j_0)$ or $(i, j) = (j_0, i_0)$ which implies our claim.

Now the desired inequality (5.24) is easy to obtain. In fact, we have that

$$\begin{aligned} & \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{F}_m} \mu_J(0) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \\ & \leq \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{F}_m} \mu_J^*(\infty) \left((|J| - 1)n - \sum_J \alpha_{ij} \right). \end{aligned}$$

Observe that $\sum_{\Theta} \alpha_{i,k+1} - n < (|J| - 1)n - \sum_J \alpha_{ij}$ for all $J \in \mathcal{F}_m$ satisfying $\Theta \subset J$ by the assumption (5.19). Then it follows from (5.39) that

$$\begin{aligned} & \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{F}_m} \mu_J^*(\infty) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \\ & \leq \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{\Theta \subset J \in \mathcal{F}_m} \mu_J^*(\infty) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) < 0. \end{aligned}$$

By Lemma 2.5, there exists a solution $\{\delta_{ij}\}$ to the system (V.1). By the induction hypothesis for $k - 1$, we have completed the proof in the first case $\sum_{i=2}^m \alpha_{i,k+1} < n$.

Now we treat the second case $\sum_{i=2}^m \alpha_{i,k+1} = n$. By Lemma 2.3, it suffices to show that the following integral is finite,

$$\int_{(B_1(0))^k} \left(\prod_S |x_i - x_j|^{-\alpha_{ij}} \right) \left(\sum_{\Theta} |x_i - x_j| \right)^{n - \sum_{\Theta} \alpha_{i,k+1}} \log \frac{C \sum_{\Theta} |x_i - x_j|}{\sum_{2 \leq i < j \leq m} |x_i - x_j|} dV_S.$$

As shown above, we can find a solution $\{\delta_{ij} : 1 \leq i < j \leq m\}$ to the system (V.1). Let $\delta_{ij} = 0$ for other (i, j) . Recall that $\sum_{\Theta} \alpha_{i,k+1} > n$. Thus there exist i_0 and j_0 with $1 \leq i_0 < j_0 \leq m$ such that $\delta_{i_0, j_0} > 0$. Choose any pair i_1 and j_1 with $2 \leq i_1 < j_1 \leq m$. For $\varepsilon > 0$, we put

$$\overline{\alpha_{ij}} = \alpha_{ij} + \delta_{ij} - \delta_i^{i_0} \delta_j^{j_0} \varepsilon + \delta_i^{i_1} \delta_j^{j_1} \varepsilon \quad (5.40)$$

for $1 \leq i < j \leq k$, where δ_s^t denotes the Kronecker symbol. It is easily verified that $\{\overline{\alpha_{ij}}\}$ satisfies the integrable conditions (5.19) with sufficiently small ε . By the induction hypothesis, the above integral converges.

The final case is $\sum_{i=2}^m \alpha_{i,k+1} > n$. The integral in (5.20) with $L_1(x_1, \dots, x_k)$ in place of the integral with respect to x_{k+1} is bounded by a constant multiple of

$$\int_{(B_1(0))^k} \left(\prod_S |x_i - x_j|^{-\alpha_{ij}} \right) \left(\sum_{\Theta} |x_i - x_j| \right)^{-\alpha_{1,k+1}} \left(\int_{\mathbb{R}^n} \prod_{i=2}^m |x_i - x_{k+1}|^{-\alpha_{i,k+1}} dx_{k+1} \right) dV_S. \quad (5.41)$$

Likewise, our task is to distribute $\alpha_{1,k+1}$ into powers $\{\alpha_{ij} : 1 \leq i < j \leq k\}$ appropriately. Then we will obtain new parameters $\{\overline{\alpha_{ij}} : 1 \leq i < j \leq k+1\}$ with $|\Theta| = m-1$. The key step is to solve the following system of linear inequalities,

$$(V.2) \begin{cases} (i) & \delta_{ij} \geq 0, \quad i < j \text{ in } \Theta; \\ (ii) & \sum_{\Theta} \delta_{ij} = \alpha_{1,k+1}; \\ (iii) & \sum_{J \cap \Theta} \delta_{ij} < (|J| - 1)n - \sum_J \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right), \quad J \in \overline{\mathcal{F}}_m; \end{cases}$$

where $\overline{\mathcal{F}}_m$ consists of all $J \subset \{1, 2, \dots, k+1\}$ with the property that J contains at least two elements in $\{1, 2, \dots, m\}$. The existence of solutions can be proved similarly. Indeed, by Lemma 2.5, for nonnegative λ_{ij} , θ_1 , θ_2 and μ_J with at least one $\mu_J > 0$ for some J in the class $\overline{\mathcal{F}}_m$ satisfying

$$\lambda_{ij} + (\theta_1 - \theta_2) - \sum_{J \ni i,j} \mu_J = 0 \quad (5.42)$$

for all $1 \leq i < j \leq m$, we need to prove

$$(\theta_1 - \theta_2) \alpha_{1,k+1} - \sum_{J \in \overline{\mathcal{F}}_m} \mu_J \left((|J| - 1)n - \sum_J \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right) \right) < 0.$$

We may assume $\theta_1 - \theta_2 = 1$. To apply the above argument in the first case, we shall make some remarks. The first observation is that the inequality (5.28) is still true with α_{ij} replaced by $\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1}$. Actually, we have

$$\sum_{s=1}^2 \sum_{J_s} \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right) \leq \sum_{J_1 \cap J_2} \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right) + \sum_{J_1 \cup J_2} \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right)$$

for all subsets J_1 and J_2 of $\{1, 2, \dots, k+1\}$.

The process described as (5.26) and (5.27) is also applicable here with \mathcal{F}_m replaced by $\overline{\mathcal{F}}_m$ in Case I and Case II there. For arbitrary nonnegative numbers $\{\mu_J(0) : J \in \overline{\mathcal{F}}_m\}$ and $\{\lambda_{ij}(0) : 1 \leq i < j \leq m\}$ satisfying equations (5.42), we can use the above argument to obtain

$$\mu_J^*(\infty) = \lim_{N_t \rightarrow \infty} \mu_J^*(N_t), \quad \lambda_{ij}^*(\infty) = \lim_{N_t \rightarrow \infty} \lambda_{ij}^*(N_t),$$

where $\{\mu_J^*(N)\}$ is obtained by one of those processes such that $\Omega_m^*(N) = \sum_{\Theta \subset J \subset \overline{\mathcal{F}}_m} \mu_J^*(N)$ is the largest over all possible processes of N steps. All symbols can be defined parallel to the first case. Similarly, we have

$$\Omega_m^*(\infty) = \lim_{N \rightarrow \infty} \Omega_m^*(N) = 1 + \min_{\Theta} \lambda_{ij}^*(\infty).$$

Note that for any J in $\overline{\mathcal{F}}_m$ satisfying $\Theta \subset J$,

$$\alpha_{1,k+1} - \left((|J| - 1)n - \sum_J (\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1}) \right) < 0.$$

Actually, this inequality is true obviously if $k+1 \in J$ by the integrable conditions (5.19). Assume $k+1 \notin J$. Then the left side of the inequality equals

$$\begin{aligned} \alpha_{1,k+1} + \sum_J \alpha_{ij} - (|J| - 1)n &= \sum_{J \cup \{k+1\}} \alpha_{ij} - \sum_{i=2}^m \alpha_{i,k+1} - (|J| - 1)n \\ &< \sum_{J \cup \{k+1\}} \alpha_{ij} - |J|n < 0 \end{aligned}$$

by $\sum_{i=2}^m \alpha_{i,k+1} > n$ and the integrable conditions.

It is clear that the integral of $\prod_{i=2}^m |x_i - x_{k+1}|^{-\alpha_{i,k+1}}$ over \mathbb{R}^n with respect to x_{k+1} is bounded by a constant multiple of the integral over $B_2(0)$ since $|x_i| \leq 1$ for $2 \leq i \leq m$. Thus the integral in (5.41) is less than a constant multiple of

$$\int_{(B_1(0))^{k+1}} \prod_{\{1,2,\dots,k+1\}} |x_i - x_j|^{-\overline{\alpha_{ij}}} dx_1 dx_2 \cdots dx_{k+1}$$

where $\overline{\alpha_{ij}} = (\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1}) + \delta_{ij}$ and $\{\delta_{ij}\}$ is a solution to the system (V.2) with $\delta_{ij} = 0$ when $i \notin \Theta$ or $j \notin \Theta$. Thus we have reduced the integral in (5.41) to an integral of form (5.20) with $|\Theta| \leq m - 1$. Since we have showed that the statement in the theorem holds when $|\Theta| = 2$, we need at most $m - 2$ steps to reduce the case $|\Theta| = m$ to the case $|\Theta| = 2$. Hence the integral (5.41) is also finite.

Until now, we have obtained the desired conclusion for \mathbf{L}_1 . For other terms \mathbf{L}_i , the argument is the same as above. Therefore the proof of the theorem is complete. \square

Remark 5.1 *The integrable criterion in the theorem is also true for Selberg integrals on the sphere S^n . More precisely, for symmetric and nonnegative exponents α_{ij} , the following Selberg integral*

$$\int_{(S^n)^{k+1}} \prod_{1 \leq i < j \leq k+1} |\xi_i - \xi_j|^{-\alpha_{ij}} d\sigma(\xi_1) d\sigma(\xi_2) \cdots d\sigma(\xi_{k+1}) < \infty$$

if and only if the integrable condition (5.19) holds. Here S^n is the unit sphere in \mathbb{R}^{n+1} and $d\sigma$ the induced Lebesgue measure on S^n . In the conformally invariant situation, by using the conformal equivalence of S^n and \mathbb{R}^n Beckner ([1]) gave explicitly the sharp constant of the multilinear functional inequality (1.4) in terms of the above Selberg integrals. In [9], Grafakos and Morpurgo calculated a three fold integral of the above form when $\alpha_{12} + \alpha_{13} + \alpha_{23} = n$. By using an analogue of Theorem 2.3 on the sphere S^n and the same argument as in the proof of Theorem 5.1, we can prove the above integrable criterion.

Now we shall establish a useful estimate by which the L^1 inequality in Theorem 1.2 follows immediately from Theorem 1.1. We state this estimate in the following theorem which will be used in the proof of Theorem 1.1 in §6.

Theorem 5.4 *Assume T , $\{\alpha_{ij} : 1 \leq i < j \leq k+1\}$, $\{p_i : 1 \leq i \leq k\}$ and $p_{k+1} = \infty$ satisfy the assumptions in Theorem 1.2. Then there exists a finite set Δ . For each $t \in \Delta$ we have nonnegative numbers $\{\beta_{ij}(t) : S\}$ such that $\{\beta_{ij}(t) : i < j \in S\}$ and $\{p_i : i \in S\}$ satisfy the system of inequalities (i), (ii) and the first type of inequalities (iii) in Theorem 1.1 and*

$$\begin{aligned} & \int_{\mathbb{R}^{n(k+1)}} \prod_{i=1}^k |f_i(x_i)| \prod_{1 \leq i < j \leq k+1} |x_i - x_j|^{-\alpha_{ij}} dx_1 dx_2 \cdots dx_{k+1} \\ & \leq C \sum_{t \in \Delta} \int_{\mathbb{R}^{nk}} \prod_{i=1}^k |f_i(x_i)| \prod_{1 \leq i < j \leq k} |x_i - x_j|^{-\beta_{ij}(t)} dx_1 dx_2 \cdots dx_k, \end{aligned} \quad (5.43)$$

where C is a constant depending only on $\alpha_{1,k+1}, \dots, \alpha_{k,k+1}$ and the dimension n .

Proof. Let Θ consist of those $i \in S$ such that $\alpha_{i,k+1} > 0$. By assumptions in Theorem 1.2 for $I = S$, it is easily seen that

$$\sum_{\Theta} \alpha_{i,k+1} = \frac{n}{p_{k+1}} + \sum_S \alpha_{i,k+1} > n.$$

To reduce the mapping properties of the $k+1$ -linear functional Λ to that of a k -linear one, we shall estimate the following integral with respect to x_{k+1} ,

$$I(x_i : i \in \Theta) = \int_{\mathbb{R}^n} \prod_{\Theta} |x_i - x_{k+1}|^{-\alpha_{i,k+1}} dx_{k+1}.$$

By the symmetry of parameters, we may assume $\Theta = \{1, 2, \dots, m\}$ with $m \geq 2$. By Lemma 2.3, we claim that the desired inequality in the theorem is still true if $I(x_i : i \in \Theta)$ is replaced by each term L_i . We first prove this statement for $m = 2$. Observe that the integral $I(x_1, x_2)$ with respect to x_{k+1} equals a constant multiple of $|x_1 - x_2|^{n-\alpha_{1,k+1}-\alpha_{2,k+1}}$. Put

$$\beta_{ij} = \alpha_{ij} + \delta_i^1 \delta_j^2 (\alpha_{1,k+1} + \alpha_{2,k+1} - n)$$

for $1 \leq i < j \leq k$. Then we see that $\{\beta_{ij} : i < j \in S\}$ and $\{p_i : i \in S\}$ satisfy (i), (ii) and the first type (a) of (iii) in Theorem 1.1. Thus our claim is valid in the case $m = 2$.

For $3 \leq m \leq k$, we claim that there exist finite families $\{\beta_{ij}(t)\}$ such that

$$\begin{aligned} & \int_{\mathbb{R}^{nk}} \prod_{i=1}^k |f_i(x_i)| \left(\prod_S |x_i - x_j|^{-\alpha_{ij}} \right) \left(\sum_{\Theta} L_i(x_1, \dots, x_m) \right) dx_1 dx_2 \cdots dx_k \\ & \leq C \sum_{t \in \Delta} \int_{\mathbb{R}^{nk}} \prod_{i=1}^k |f_i(x_i)| \prod_S |x_i - x_j|^{-\beta_{ij}(t)} dx_1 dx_2 \cdots dx_k, \end{aligned} \quad (5.44)$$

where Δ is a finite set and $\{\beta_{ij}(t)\}$ and $\{1/p_i\}$ satisfy conditions (i), (ii) and (a) of (iii) in Theorem 1.1. Since arguments for different L_i 's are the same, we need only establish the desired

estimate concerning \mathbf{L}_1 . There are three possible cases for \mathbf{L}_1 . The treatment of two previous cases $\sum_{i=2}^m \alpha_{i,k+1} \leq n$ will be reduced to the following system of linear inequalities:

$$(V.3) \begin{cases} (i) & \delta_{ij} \geq 0, \quad 1 \leq i < j \leq m; \\ (ii) & \sum_{\Theta} \delta_{ij} = \sum_{\Theta} \alpha_{i,k+1} - n; \\ (iii) & \sum_{J \cap \Theta} \delta_{ij} < \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) \quad \text{for } J \in \mathcal{F}_m \text{ and } J \neq S; \\ (iv) & \sum_{S \cap \Theta} \delta_{ij} \leq \left((k - 1)n - \sum_S \alpha_{ij} \right) \wedge \left(kn - \sum_S \alpha_{ij} - \sum_S \frac{n}{p_i} \right); \end{cases}$$

where \mathcal{F}_m is the class of all subsets J of S which contains at least two members in $\{1, 2, \dots, m\}$. Indeed, the inequality (iv) above is an equality. By (ii) in the system (V.3), this observation follows from $\sum_{S \cup \{k+1\}} \alpha_{ij} < kn$ and the condition (i) in Theorem 1.1. For this reason, the system (V.3) is equivalent to the system of inequalities consisting of (i), (ii) and (iii). We add (iv) for convenience and its usefulness will be seen in the following argument. As above, the proof of existence of a solution relies on Lemma 2.5. Assume λ_{ij} , θ_1 , θ_2 and μ_J are nonnegative numbers satisfying

$$\lambda_{ij} - \sum_{J \ni i,j} \mu_J + (\theta_1 - \theta_2) = 0, \quad 1 \leq i < j \leq m. \quad (5.45)$$

Here there is at least one $\mu_J > 0$ for some proper subset J of S in the class \mathcal{F}_m . Under these conditions, we have to prove

$$\begin{aligned} & (\theta_1 - \theta_2) \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) \\ & < \sum_{J \in \mathcal{F}_m} \mu_J \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right). \end{aligned} \quad (5.46)$$

This inequality is obviously true if $\theta_1 - \theta_2 \leq 0$ since $\mu_J > 0$ for some proper subset $J \in \mathcal{F}_m$ of S . Thus it suffices to show the inequality for $\theta_1 - \theta_2 > 0$. By dilation, we may assume $\theta_1 - \theta_2 = 1$. There is an important observation like (5.28) given as follows:

$$\begin{aligned} & \left((|J_1 \cap J_2| - 1)n - \sum_{J_1 \cap J_2} \alpha_{ij} \right) \wedge \left(|J_1 \cap J_2|n - \sum_{J_1 \cap J_2} \alpha_{ij} - \sum_{J_1 \cap J_2} \frac{n}{p_i} \right) \\ & + \left((|J_1 \cup J_2| - 1)n - \sum_{J_1 \cup J_2} \alpha_{ij} \right) \wedge \left(|J_1 \cup J_2|n - \sum_{J_1 \cup J_2} \alpha_{ij} - \sum_{J_1 \cup J_2} \frac{n}{p_i} \right) \\ & \leq \sum_{s=1}^2 \left\{ \left((|J_s| - 1)n - \sum_{J_s} \alpha_{ij} \right) \wedge \left(|J_s|n - \sum_{J_s} \alpha_{ij} - \sum_{J_s} \frac{n}{p_i} \right) \right\} \end{aligned} \quad (5.47)$$

for all subsets J_1 and J_2 of S . It suffices to show this inequality in all possible cases. First observe that (5.28) becomes an equality if and only if

$$\alpha_{ij} = 0, \quad (i, j) \in \{(s, t) : s < t \in J_1 \cup J_2\} - \bigcup_{i=1}^2 \{(s, t) : s < t \in J_i\}. \quad (5.48)$$

It is also true that

$$\begin{aligned} & \left(|J_1 \cap J_2|n - \sum_{J_1 \cap J_2} \alpha_{ij} - \sum_{J_1 \cap J_2} \frac{n}{p_i} \right) + \left(|J_1 \cup J_2|n - \sum_{J_1 \cup J_2} \alpha_{ij} - \sum_{J_1 \cup J_2} \frac{n}{p_i} \right) \\ & \leq \sum_{s=1}^2 \left(|J_s|n - \sum_{J_s} \alpha_{ij} - \sum_{J_s} \frac{n}{p_i} \right), \end{aligned}$$

where equality is valid if and only if (5.48) holds. Similarly we also have

$$\begin{aligned} & \left((|J_1 \cap J_2| - 1)n - \sum_{J_1 \cap J_2} \alpha_{ij} \right) + \left(|J_1 \cup J_2|n - \sum_{J_1 \cup J_2} \alpha_{ij} - \sum_{J_1 \cup J_2} \frac{n}{p_i} \right) \\ & \leq \left((|J_1| - 1)n - \sum_{J_1} \alpha_{ij} \right) + \left(|J_2|n - \sum_{J_2} \alpha_{ij} - \sum_{J_2} \frac{n}{p_i} \right) \end{aligned}$$

and this inequality becomes an equality if and only if (5.48) holds and $p_i = \infty$ for i in J_1 but not in J_2 . The remaining case of (5.47) can be proved similarly. Combing above results, we see that if (5.47) becomes an equality then we have (5.48). This fact will be used later.

To show the inequality (5.46), we also need the process in (5.26) and (5.27). By (5.47), an analogue of (5.29) in the present situation can be written as follow,

$$\begin{aligned} & \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{F}_m} \mu_J(N-1) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) \\ & \leq \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{F}_m} \mu_J(N) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) \quad (5.49) \end{aligned}$$

where $\mu_J(N)$ are obtained from $\mu_J(N-1)$ by one step described in (5.26) and (5.27).

We shall take the same notations as in the proof of Theorem 5.1 in the following. Let $\{\mu_J^*(N)\}$ be obtained by one of processes consisting of N steps such that $\Omega_m^*(N)$ is the largest over all these processes. By a similar argument in the proof of Theorem 5.1, for each $N \geq 1$ we can obtain $\{\mu_J^*(N)\}$ and $\{\lambda_{ij}^*(N)\}$ from initially given nonnegative $\{\mu_J(0)\}$ and $\{\lambda_{ij}(0)\}$ satisfying (5.45) with $\theta_1 - \theta_2 = 1$. By passing to a subsequence $\{N_t\}$, we also use the notations $\mu_J^*(\infty)$ and $\lambda_{ij}^*(\infty)$ to denote the limits of $\mu_J^*(N_t)$ and $\lambda_{ij}^*(N_t)$ as $N_t \rightarrow \infty$ respectively. It is clear that (5.39) is also true and hence $\Omega_m^*(\infty) \geq 1$. The argument to follow is somewhat different depending on whether $\mu_S^*(\infty) = 1$ and $\mu_J^*(\infty) = 0$ for all proper subsets J of S in \mathcal{F}_m . Observe that

$$\sum_{\Theta} \alpha_{i,k+1} - n = \left((|S| - 1)n - \sum_S \alpha_{ij} \right) \wedge \left(|S|n - \sum_S \alpha_{ij} - \sum_S \frac{n}{p_i} \right)$$

and the right side equals $|S|n - \sum_S \alpha_{ij} - \sum_S n/p_i$. And we also have

$$\sum_{\Theta} \alpha_{i,k+1} - n < \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right)$$

for all proper subsets J of S satisfying $\Theta \subset J$. By (5.49), we have

$$\begin{aligned} & - \sum_{J \in \mathcal{F}_m} \mu_J(0) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) \\ & \leq - \sum_{J \in \mathcal{F}_m} \mu_J^*(\infty) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right). \end{aligned}$$

To simplify the notations, let $\mathbf{H}(\theta_1 - \theta_2, \lambda_{ij}, \mu_J)$ be the object function

$$(\theta_1 - \theta_2) \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{F}_m} \mu_J \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \wedge \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right).$$

Since $\theta_1 - \theta_2 = 1$, we also use $\mathbf{H}(\lambda_{ij}, \mu_J)$ instead of $\mathbf{H}(\theta_1 - \theta_2, \lambda_{ij}, \mu_J)$ for simplicity. If $\Omega_m^*(\infty) > 1$, then the inequality (5.46) follows immediately. Indeed,

$$\mathbf{H}(\lambda_{ij}(0), \mu_J(0)) \leq \mathbf{H}(\lambda_{ij}^*(\infty), \mu_J^*(\infty))$$

and

$$\mathbf{H}(\lambda_{ij}^*(0), \mu_J^*(0)) < \Omega_m^*(\infty) \left(\sum_{\Theta} \alpha_{i,k+1} - n \right) - \sum_{J \in \mathcal{C}_m(\infty)} \mu_J^*(\infty) \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \leq 0.$$

Similarly, $\mathbf{H}(\lambda_{ij}(0), \mu_J(0)) < 0$ if $\Omega_m^*(\infty) = 1$ and $\mu_J^*(\infty) > 0$ for at least one proper subset J of S in \mathcal{F}_m . Combing results in these two cases, we have proved that

$$\mathbf{H}(\lambda_{ij}, \mu_J) < 0 \quad (5.50)$$

for all nonnegative λ_{ij} and μ_J satisfying $\sum_{J \ni i,j} \mu_J = 1 + \lambda_{ij}$ with some $\lambda_{ij} > 0$.

The most delicate case is $\Omega_m^*(\infty) = 1$ and $\mu_J^*(\infty) = 0$ for all proper subsets $J \in \mathcal{F}_m$ of S . By the definition of $\Omega_m^*(\infty)$, we see that $\mu_S^*(\infty) = 1$ and $\mathbf{H}(\lambda_{ij}^*(\infty), \mu_J^*(\infty)) = 0$. The above argument should be modified slightly. In this case, note that all $\lambda_{ij}^*(\infty) = 0$ and hence $\lambda_{ij}(0) = 0$ because $\lambda_{ij}(N)$ does not decrease in a continuous process. Then the system (5.45) becomes

$$\sum_{J \ni i,j} \mu_J(0) = 1, \quad 1 \leq i < j \leq m. \quad (5.51)$$

Though we have not proven $\mathbf{H}(\lambda_{ij}(0), \mu_J(0)) < 0$ yet, a weak form of this inequality is true. For all nonnegative λ_{ij} and μ_J satisfying the equations (5.45) with $\theta_1 - \theta_2 = 1$,

$$\begin{aligned} \mathbf{H}(\lambda_{ij}(0), \mu_J(0)) &\leq \mathbf{H}(\lambda_{ij}^*(\infty), \mu_J^*(\infty)) \\ &\leq \mathbf{H}(\lambda_{ij} = 0, \mu_J = \delta_S^J) = 0, \end{aligned} \quad (5.52)$$

where $\delta_S^J = 1$ if $J = S$ and $\delta_S^J = 0$ if $J \neq S$.

We now treat the case in which all $\lambda_{ij}(0) = 0$. Let \mathcal{A}_m , \mathcal{B}_m and \mathcal{C}_m be defined as in (5.30). We shall divide this present situation into two cases. The first case is $\Omega_m(0) = 1$. Recall $\Omega_m(0) = \sum_{J \in \mathcal{C}_m(0)} \mu_J(0)$. By equations (5.51), we have $\mu_J(0) = 0$ for all $J \in \mathcal{B}_m(0)$. The choice of $\{\mu_J(0)\}$ implies the existence of $\mu_J(0) > 0$ for some proper subset J of S in the class $\mathcal{C}_m(0)$. Thus

$$\mathbf{H}(\lambda_{ij}(0) = 0, \mu_J(0)) < 0.$$

We now consider the second case $\Omega_m(0) < 1$. This implies that there exists a positive $\mu_J(0)$ for $J \in \mathcal{B}_m(0)$. Now we claim that there exist J_1 and J_2 in $\mathcal{B}_m(0)$ with nonempty intersection such that

$$J_1 \cap \Theta \not\subset J_2 \cap \Theta \quad \text{and} \quad J_2 \cap \Theta \not\subset J_1 \cap \Theta. \quad (5.53)$$

Assume the converse. Then either $J_1 \cap \Theta \subset J_2 \cap \Theta$ or $J_2 \cap \Theta \subset J_1 \cap \Theta$ is true for all J_1 and J_2 in $\mathcal{B}_m(0)$ with $J_1 \cap J_2 \neq \emptyset$. Let $J_0 \in \mathcal{B}_m(0)$ satisfy

$$|J_0 \cap \Theta| = \max_{J \in \mathcal{B}_m(0)} |J \cap \Theta|.$$

We shall use the choice of J_0 to derive a contradiction. Since $J_0 \cap \Theta$ is a proper subset of Θ , we may choose $i_0 \in \Theta$ but $i_0 \notin J_0 \cap \Theta$. Choose a $j_0 \in J_0 \cap \Theta$ arbitrarily. Then by the choice of J_0 , we see that all $\mu_J(0) = 0$ for those J in $\mathcal{B}_m(0)$ satisfying $i_0, j_0 \in J$. Then we have $\Omega_m(0) = 1$ by the equation in (5.51) with $(i, j) = (i_0, j_0)$ or $(i, j) = (j_0, i_0)$. As a consequence, $\mathcal{B}_m(0)$ becomes a empty class which contradicts our assumption. Choose $J_1, J_2 \in \mathcal{B}_m(0)$ with $J_1 \cap J_2 \neq \emptyset$ satisfy

$$\begin{aligned} J_1 \cap \{1, 2, \dots, m\} &\not\subseteq J_2 \cap \{1, 2, \dots, m\} \\ J_2 \cap \{1, 2, \dots, m\} &\not\subseteq J_1 \cap \{1, 2, \dots, m\}. \end{aligned}$$

Then we apply the recursion (5.26) or (5.27) to J_1 and J_2 and then obtain $\{\mu_J(1)\}$ and $\{\lambda_{ij}(1)\}$. By this process, we will obtain at least one $\lambda_{ij}(1) > 0$. Indeed, we may choose

$$\begin{aligned} i \in J_1 \cap \{1, 2, \dots, m\} \quad \text{but} \quad i \notin J_2 \cap \{1, 2, \dots, m\} \\ j \in J_2 \cap \{1, 2, \dots, m\} \quad \text{but} \quad j \notin J_1 \cap \{1, 2, \dots, m\}. \end{aligned}$$

It follows that $\lambda_{ij}(1) = \mu_{J_1}(0) \wedge \mu_{J_2}(0) > 0$ since $J_1, J_2 \in \mathcal{A}_m(0)$. Since the object function $\mathbf{H}(\lambda_{ij}(N), \mu_J(N))$ increase as N , we see that

$$\mathbf{H}(\lambda_{ij}(0) = 0, \mu_J(0)) \leq \mathbf{H}(\lambda_{ij}(1), \mu_J(1)) < 0.$$

Here $\mathbf{H}(\lambda_{ij}(1), \mu_J(1)) < 0$ is a consequence of (5.50). Therefore there exists at least one solution to the system (V.3).

Let $\{\delta_{ij}\}$ be a solution to the system (V.3). Here we also put $\delta_{ij} = 0$ if either i or j not in Θ . If $\sum_{i=2}^m \alpha_{i,k+1} < n$, then our claim (5.44) is true by setting $\beta_{ij} = \alpha_{ij} + \delta_{ij}$. In the case $\sum_{i=2}^k \alpha_{i,k+1} = n$, the treatment is similar as (5.40). First observe that there exists some $\delta_{i_0 j_0} > 0$ for $1 \leq i_0 < j_0 \leq m$. Choose a pair (i_1, j_1) in $\{2, \dots, m\}$ arbitrarily. Set

$$\beta_{ij} = \alpha_{ij} + \delta_{ij} - \delta_i^{i_0} \delta_j^{j_0} \varepsilon + \delta_i^{i_1} \delta_j^{j_1} \varepsilon \quad (5.54)$$

for all $1 \leq i < j \leq k$. Then $\{\beta_{ij}\}$ satisfies the desired estimate with sufficiently small $\varepsilon > 0$.

Now we turn our attention to the final case $\sum_{i=2}^m \alpha_{i,k+1} > n$. Then \mathbf{L}_1 equals

$$\left(\sum_{\Theta} |x_i - x_j| \right)^{-\alpha_{1,k+1}} \int_{\mathbb{R}^n} \prod_{i=2}^m |x_i - x_{k+1}|^{-\alpha_{i,k+1}} dx_{k+1}.$$

The system (V.3) is not applicable in this situation. We shall adapt the treatments of the system (V.2). The corresponding system of linear inequalities is given as follows:

$$(V.4) \left\{ \begin{array}{ll} (i) & \delta_{ij} \geq 0, \quad 1 \leq i < j \leq m; \\ (ii) & \sum_{\Theta} \delta_{ij} = \alpha_{1,k+1}; \\ (iii) & \sum_{J \cap \Theta} \delta_{ij} < \left((|J| - 1)n - \mathbf{B}_J \right) \wedge \left(|J|n - \mathbf{B}_J - \sum_J \frac{n}{p_i} \right), \quad J \in \overline{\mathcal{F}}_m, \quad J \neq S \cup \{k+1\}; \\ (iv) & \sum_{S \cup \{k+1\} \cap \Theta} \delta_{ij} \leq (kn - \mathbf{B}_{S \cup \{k+1\}}) \wedge \left((k+1)n - \mathbf{B}_{S \cup \{k+1\}} - \sum_{S \cup \{k+1\}} \frac{n}{p_i} \right); \end{array} \right.$$

where $\overline{\mathcal{F}}_m$ is the class of all subsets $J \subset \{1, 2, \dots, k+1\}$ with $|J \cap \{1, 2, \dots, m\}| \geq 2$ and B_J is given by, for each $J \in \overline{\mathcal{F}}_m$,

$$B_J = \sum_J \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right).$$

The existence of a solution to the system can be proved as above. The outline of argument can be presented as follows. For nonnegative λ_{ij} , μ_J , θ_1 and θ_2 satisfying

$$\lambda_{ij} - \sum_{J \ni i, j} \mu_J + (\theta_1 - \theta_2) = 0, \quad 1 \leq i < j \leq m,$$

where there exists one $\mu_J > 0$ for some proper subset J of $S \cup \{k+1\}$ in the class $\overline{\mathcal{F}}_m$, it is enough to show that

$$(\theta_1 - \theta_2) \alpha_{1,k+1} - \sum_{J \in \overline{\mathcal{F}}_m} \mu_J \left((|J| - 1)n - B_J \right) \wedge \left(|J|n - B_J - \sum_J \frac{n}{p_i} \right) < 0. \quad (5.55)$$

The above inequality is obvious for $\theta_1 - \theta_2 \leq 0$. By scaling, we may assume $\theta_1 - \theta_2 = 1$. In this setting, the argument is the same as in the proof of existence of solutions to the system (V.2). We omit the details here.

Put $\delta_{ij} = 0$ for $i \notin \Theta$ or $j \notin \Theta$. Let $\{\delta_{ij} : 1 \leq i < j \leq m\}$ be a solution to the system (V.4). Set

$$\overline{\alpha}_{ij} = (\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1}) + \delta_{ij}$$

for $1 \leq i < j \leq k+1$. Then the left side integral in (5.44) with L_1 in place of $\sum_{\Theta} L_i$ is bounded by a constant multiple of

$$\int_{\mathbb{R}^{n(k+1)}} \prod_{i=1}^k |f_i(x_i)| \prod_{1 \leq i < j \leq k+1} |x_i - x_j|^{-\overline{\alpha}_{ij}} dx_1 dx_2 \cdots dx_{k+1}$$

which reduces $|\Theta| = m$ to $|\Theta| = m - 1$.

Repeating the above argument finite times, we will obtain the desired inequality in the theorem. \square

6 Proof of the main results

In this part, we shall give a complete proof of the main results. At the beginning, we shall make some observations which turn out to be useful for interpolations. As shown in Theorem 5.1, the integral in (1.3) is absolutely convergent for arbitrary $f_i \in C_0^\infty$ if and only if $\sum_J \alpha_{ij} < (|J| - 1)n$ for all subsets J of $\{1, 2, \dots, k+1\}$ with $|J| \geq 2$. The existence of such nonnegative numbers is obvious. For any given $\{\alpha_{ij}\}$ satisfying these integrable conditions, a natural question arises whether there is a set of positive numbers $\{p_i\}$ such that $\{\alpha_{ij}\}$ and $\{p_i\}$ satisfy conditions (i), (ii) and (iii) in Theorem 1.1. The answer is affirmative except some trivial cases. For example, the simplest case all $\alpha_{ij} = 0$ should be left out since the boundedness of Λ is valid only if all $p_i = 1$. Moreover, we may further assume that for any given i not all α_{ij} are zero with j ranging over $\{1, 2, \dots, k+1\}$. In this setting, the existence of $\{p_i\}$ can be stated as follows.

Theorem 6.1 Assume $\alpha_{ij} \geq 0$ satisfy the inequalities (ii) in Theorem 1.1 and

$$\sum_{i \in J} \sum_{j \in J^c} \alpha_{ij} > 0 \quad (6.56)$$

for any nonempty and proper subset J of $\{1, 2, \dots, k+1\}$. Then there exist infinitely many $\{p_i\}$ such that

$$(VI.1) \begin{cases} (i) & 1 < p_i < \infty, \quad 1 \leq i \leq k+1; \\ (ii) & \sum_{i=1}^{k+1} \frac{1}{p_i} + \sum_{1 \leq i < j \leq k+1} \frac{\alpha_{ij}}{n} = k+1; \\ (iii) & \sum_J \frac{1}{p_i} + \sum_J \frac{\alpha_{ij}}{n} < |J|, \quad J \neq \emptyset, \quad J \subsetneq \{1, 2, \dots, k+1\}. \end{cases}$$

Proof. We begin with discussing the necessity of the additional assumption (6.56) which does not lose generality. This assumption is only necessary to ensure the existence of a solution to the system (VI.1). For general $\{\alpha_{ij}\}$ satisfying $\sum_J \alpha_{ij} < (|J| - 1)n$ for all subsets J with $|J| \geq 2$, we only need assume that for each i there exist $\alpha_{ij} > 0$ with j ranging over $S \cup \{k+1\}$. By this weaker assumption, we can divide $\{1, 2, \dots, k+1\}$ into disjoint subsets J_1, J_2, \dots, J_l with $|J_i| \geq 2$ such that α_{ij} is positive with $i < j$ only if

$$(i, j) \in \bigcup_{u=1}^l \{(s, t) : s < t, s, t \in J_u\}.$$

If each J_i can not be decomposed further as above, i.e., $\sum_{u \in I} \sum_{v \in J_i - I} \alpha_{uv} > 0$ for any nonempty and proper subset I of J_i , then we shall reduce matters to l multilinear functionals $\{\Lambda_{J_i}\}$ of form (1.3). Therefore we may further assume that $\{1, 2, \dots, k+1\}$ cannot be decomposed as above. This is equivalent to $\sum_{i \in J} \sum_{j \in J^c} \alpha_{ij} > 0$ for all nonempty and proper subsets J or $\{1, 2, \dots, k+1\}$. Thus the additional assumption (6.56) does not lose generality.

Now we turn to verify the existence of $\{p_i\}$. Define $\delta_i = 1/p_i$. Then $\{\delta_i\}$ satisfies the following system of linear inequalities:

$$(VI.2) \begin{cases} (i) & \sum_{i=1}^{k+1} \delta_i = k+1 - \sum_{1 \leq i < j \leq k+1} \frac{\alpha_{ij}}{n}; \\ (ii) & \delta_i > 0, \quad 1 \leq i \leq k+1; \\ (iii) & \sum_J \delta_i < |J| - \sum_J \frac{\alpha_{ij}}{n}, \quad J \neq \emptyset, \quad J \subsetneq \{1, 2, \dots, k+1\}; \\ (iv) & \sum_{\{1, 2, \dots, k+1\}} \delta_i \leq k+1 - \sum_{\{1, 2, \dots, k+1\}} \frac{\alpha_{ij}}{n}. \end{cases}$$

If we take $J = \{i\}$, then (iii) implies $\delta_i < 1$. Assume $\theta_1, \theta_2, \pi_i, \mu_J$ are nonnegative numbers satisfying

$$(\theta_1 - \theta_2) + \pi_i - \sum_{J \ni i} \mu_J = 0, \quad 1 \leq i \leq k+1 \quad (6.57)$$

where either $\pi_i > 0$ for some i or $\mu_J > 0$ for some nonempty and proper subset J . By Lemma 2.5, it is enough to show

$$(\theta_1 - \theta_2) \left(k+1 - \sum_{1 \leq i < j \leq k+1} \frac{\alpha_{ij}}{n} \right) - \sum_J \mu_J \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) < 0. \quad (6.58)$$

Assume $\theta_1 - \theta_2 = 1$. For any initial data $\{\mu_J(0)\}$ and $\{\pi_i(0)\}$ satisfying $\sum_{J \ni i} \mu_J = 1 + \pi_i$, we adapt the recursion (5.26) and (5.27) to the following variant. For nonempty subsets J_1 and J_2 with $\mu_{J_1}(N-1)\mu_{J_2}(N-1) > 0$, we put

$$\begin{cases} \mu_{J_t}(N) = \mu_{J_t}(N-1) - \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1), & t = 1, 2. \\ \mu_{J_1 \cap J_2}(N) = \mu_{J_1 \cap J_2}(N-1) + \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) & \text{if } J_1 \cap J_2 \neq \emptyset \\ \mu_{J_1 \cup J_2}(N) = \mu_{J_1 \cup J_2}(N-1) + \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \end{cases} \quad (6.59)$$

and $\mu_J(N) = \mu_J(N-1)$ for other nonempty subsets J of $\{1, 2, \dots, k+1\}$. Here we apply this process to all nonempty subsets J_1 and J_2 without the assumption $J_1 \cap J_2 \neq \emptyset$. The process does not change the value of π_i , i.e., $\pi_i(N) = \pi_i(0)$, and equations (6.57) are still true with $\{\mu_J(0)\}$ replaced by $\{\mu_J(N)\}$. Let $\mu_{S \cup \{k+1\}}^*(N)$ be the supremum of $\mu_{S \cup \{k+1\}}(N)$ among all possible N continuous steps of the process. Let $\{\mu_J^*(N)\}$ be obtained by one of those N continuous steps by which $\mu_{S \cup \{k+1\}}^*(N)$ can be obtained. Then $\mu_{S \cup \{k+1\}}^*(N)$ increases as N and $\{\mu_J^*(N)\}$ has a uniform upper bound for each nonempty subset J . By passing to a subsequence $\{N_t\}$, we obtain Cauchy sequences $\{\mu_J^*(N_t)\}$. Let

$$\mu_J^*(\infty) = \lim_{t \rightarrow \infty} \mu_J^*(N_t), \quad J \subset S \cup \{k+1\}. \quad (6.60)$$

By a similar argument in §5, we can show that $\mu_{S \cup \{k+1\}}^*(\infty)$ is stable. Indeed, the union of all proper subsets J satisfying $\mu_J^*(\infty) > 0$ is a proper subset of $S \cup \{k+1\}$. Thus we get

$$\mu_{S \cup \{k+1\}}^*(\infty) = 1 + \min_{1 \leq i \leq k+1} \pi_i. \quad (6.61)$$

Notice that the object function in (6.58) increases as N with μ_J replaced by $\mu_J(N)$. For this reason, we see that the inequality (6.58) is true when there exists a positive π_i . Indeed, if some π_i is positive and $\mu_{S \cup \{k+1\}}^*(\infty) = 1$, then we obtain a $\mu_J^*(\infty) > 0$ for some nonempty $J \subsetneq S \cup \{k+1\}$.

Assume now all π_i are zero. In this case, the argument is somewhat different and the additional assumption (6.56) will be used. By our choice of the parameters in (6.57), it follows that there is a positive $\mu_J = \mu_J(0)$ with J being a nonempty and proper subset of $S \cup \{k+1\}$. Thus $\mu_{S \cup \{k+1\}}(0) < 1$. For all N , it is true that

$$\sum_{J \ni i} \mu_J^*(N) = 1, \quad 1 \leq i \leq k+1.$$

Recall that $\mu_{S \cup \{k+1\}}^*(N)$ tends to 1. This implies that each sequence $\mu_J^*(N)$ tends to zero for all proper subsets J . Choose a sufficiently large N_0 such that

$$\mu_{S \cup \{k+1\}}^*(N_0) > 1 - \varepsilon > \mu_{S \cup \{k+1\}}(0)$$

with $\varepsilon > 0$ small enough. This observation shows that in the process by which we obtain $\{\mu_J^*(N_0)\}$, consisting of N_0 continuous steps, there is a M -th step such that $\mu_{S \cup \{k+1\}}(M)$ is strictly larger than $\mu_{S \cup \{k+1\}}(M-1)$. Combining the additional assumption (6.56) together with the observation (5.48), we obtain

$$\sum_J \mu_J(M) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) < \sum_J \mu_J(M-1) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right).$$

The following argument may vary depending on whether $\mu_{S \cup \{k+1\}}(N_0) = 1$. If $\mu_{S \cup \{k+1\}}(N_0)$ were equal to 1, we would obtain, using the notation $A_k = k + 1 - \sum_{S \cup \{k+1\}} \alpha_{ij}/n$,

$$\begin{aligned} A_k - \sum_J \mu_J(0) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) &\leq A_k - \sum_J \mu_J(M-1) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) \\ &< A_k - \sum_J \mu_J(M) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right). \end{aligned}$$

Notice that we also have

$$A_k - \sum_J \mu_J(M) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) \leq A_k - \sum_J \mu_J(N_0) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right),$$

where the process used to obtain $\mu_J(N_0)$ is one of those N continuous steps such that $\mu_{S \cup \{k+1\}}(N_0) = \mu_{S \cup \{k+1\}}^*(N_0)$ and $\mu_{S \cup \{k+1\}}(M) > \mu_{S \cup \{k+1\}}(M-1)$. The desired inequality (6.58) follows from the assumption $\mu_{S \cup \{k+1\}}(N_0) = 1$.

If $\mu_{S \cup \{k+1\}}(N_0) < 1$, we only have

$$A_k - \sum_J \mu_J(0) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) < A_k - \sum_J \mu_J(N_0) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right).$$

However, we may regard $\{\mu_J^*(N_0)\}$ as new initial data satisfying the system of equations (6.57) with all $\pi_i = 0$ and $\theta_1 - \theta_2 = 1$ since there exist at least two positive $\mu_J^*(N_0)$ for nonempty and proper subsets J . This implies

$$A_k - \sum_J \mu_J(N_0) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) = A_k - \sum_J \mu_J^*(N_0) \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) \leq 0.$$

Combing above results, we have completed the proof of (6.58). Thus the system (VI.2) has a solution.

It remains to show that there are infinitely many solutions to the system (VI.2). Let \mathbb{P} be the hyperplane in \mathbb{R}^{k+1} given by

$$\sum_{i=1}^{k+1} x_i = k + 1 - \sum_{1 \leq i < j \leq k+1} \frac{\alpha_{ij}}{n} \quad (6.62)$$

for $x = (x_1, \dots, x_{k+1})$ in \mathbb{R}^{k+1} . Give \mathbb{P} the subset topology of \mathbb{R}^{k+1} . Then the set of solutions of the system (VI.2) forms an open and convex subset of \mathbb{P} . Let $\{\delta_i^{(1)}\}$ and $\{\delta_i^{(2)}\}$ be two solutions. Then it is clear that their convex combinations $\{\lambda \delta_i^{(1)} + (1 - \lambda) \delta_i^{(2)}\}$ are also solutions for all $0 < \lambda < 1$. Assume $\{\delta_i\}$ is a given solution to the system (VI.2). With sufficiently small $\varepsilon > 0$, all points in the ε -neighborhood of $\{\delta_i\}$ in \mathbb{P} are also solutions. Indeed, if $\rho = (\rho_1, \dots, \rho_{k+1}) \in \mathbb{P}$ and $(\sum_{i=1}^{k+1} |\rho_i - \delta_i|^2)^{1/2} < \varepsilon$, then the inequalities (ii) and (iii) are also true for $\{\rho_i\}$ with $\varepsilon > 0$ sufficiently small. Thus we have established our claim. The proof is therefore concluded. \square

We shall now extend Theorem 4.1 to a general result which is useful for multilinear interpolation.

Theorem 6.2 Assume $\alpha_{ij} \geq 0$ for $1 \leq i < j \leq k+1$ and $1 < p_i < \infty$ for $1 \leq i \leq k+1$ satisfy the conditions (i), (ii) in Theorem 1.1 and

$$\sum_J \alpha_{ij} + \sum_J \frac{n}{p_i} < |J|n$$

for all nonempty and proper subsets J of $\{1, 2, \dots, k+1\}$. Then it is true that

$$\int_{\mathbb{R}^{nk}} \prod_{1 \leq i < j \leq k+1} |x_i - x_j|^{-\alpha_{ij}} \prod_{i=2}^{k+1} |x_i|^{-n/p_i} dx_2 \cdots dx_{k+1} \leq C|x_1|^{-n(1-1/p_1)} \quad (6.63)$$

with the constant C independent of x_1 .

Proof. Assume the above estimate is true for $k-1$ with $k \geq 2$. Now we shall prove that it is also true for k . By Lemma 2.3, we have

$$\int_{\mathbb{R}^n} \prod_S |x_i - x_{k+1}|^{-\alpha_{i,k+1}} |x_{k+1}|^{-n/p_{k+1}} dx_{k+1} \leq C \sum_{\Theta} \mathbf{L}_i(x_1, \dots, x_k) + \mathbf{L}_{k+1}(x_1, \dots, x_k)$$

with Θ consisting of all those $i \in S$ such that $\alpha_{i,k+1} > 0$. If Θ consists of a single point, then it is easy to see that the desired estimate can be reduced to the case $k-1$. Indeed, for example, we take $\Theta = \{1\}$. Put

$$\overline{\alpha_{ij}} = \alpha_{ij}, \quad \frac{1}{\overline{p_i}} = \frac{1}{p_i} + \delta_i^1 \left(\frac{\alpha_{1,k+1}}{n} + \frac{1}{p_{k+1}} - 1 \right).$$

Then it is easy to verify that $\{\overline{\alpha_{ij}}\}$ and $\{\overline{p_i}\}$ satisfy assumptions in the case $k-1$. By the induction hypothesis, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{nk}} \prod_{1 \leq i < j \leq k+1} |x_i - x_j|^{-\alpha_{ij}} \prod_{i=2}^{k+1} |x_i|^{-n/p_i} dx_2 \cdots dx_{k+1} \\ & \leq C|x_1|^{-n/p_{k+1} - \alpha_{1,k+1} + n} \int_{\mathbb{R}^{n(k-1)}} \prod_S |x_i - x_j|^{-\overline{\alpha_{ij}}} \prod_{i=2}^k |x_i|^{-n/\overline{p_i}} dx_2 \cdots dx_k \\ & \leq C|x_1|^{-n/p_{k+1} - \alpha_{1,k+1} + n} |x_1|^{-n(1-1/\overline{p_1})}, \end{aligned}$$

where the last term is just equal to $C|x_1|^{-n(1-1/p_1)}$.

Assume $|\Theta| = m$ with $2 \leq m \leq k$. By the symmetry of parameters, we may assume $\Theta = \{1, 2, \dots, m\}$.

Case 1: $\sum_{i=2}^k \alpha_{i,k+1} + n/p_{k+1} < n$.

In this case,

$$\mathbf{L}_1(x_1, \dots, x_k) = \left(\sum_{\Theta} |x_i - x_j| + \sum_{\Theta} |x_i| \right)^{n - \sum_{\Theta} \alpha_{i,k+1} - n/p_{k+1}}.$$

To use the induction hypothesis, we need to solve the following system of linear inequalities:

$$(VI.3) \begin{cases} (i) & \delta_{ij} \geq 0, \delta_i \geq 0 \quad \text{in } \Theta; \\ (ii) & \sum_{\Theta} \delta_{ij} + n \sum_{\Theta} \delta_i = \sum_{\Theta} \alpha_{i,k+1} + n/p_{k+1} - n; \\ (iii) & \sum_{J \cap \Theta} \delta_{ij} < (|J| - 1)n - \sum_J \alpha_{ij}, \quad J \in \mathcal{F}_{\Theta}; \\ (iv) & \sum_{J \cap \Theta} \delta_{ij} + n \sum_{J \cap \Theta} \delta_i < |J|n - \sum_J \alpha_{ij} - \sum_J n/p_i, \quad J \in \mathcal{G}_{\Theta}, \quad J \neq S, \end{cases}$$

where \mathcal{F}_Θ consists of all subsets J of S with $|J \cap \Theta| \geq 2$ and \mathcal{G}_Θ is the class of all subsets J of S with $|J \cap \Theta| \geq 1$. Assume π_{ij} , π_i , θ_1 , θ_2 , μ_J and γ_J are nonnegative numbers satisfying

$$\begin{cases} \pi_{ij} + (\theta_1 - \theta_2) - \sum_{J \ni i, j} \mu_J - \sum_{J \ni i, j} \gamma_J = 0, & i < j \in \Theta; \\ \pi_i + n(\theta_1 - \theta_2) - n \sum_{J \ni i} \gamma_J = 0, & i \in \Theta. \end{cases} \quad (6.64)$$

Here there exists at least one positive number in the union of $\{\mu_J : J \in \mathcal{F}_\Theta\}$ and $\{\gamma_J : J \in \mathcal{G}_\Theta, J \neq S\}$. We shall prove

$$\begin{aligned} & (\theta_1 - \theta_2) \left(\sum_{\Theta} \alpha_{i, k+1} + \frac{n}{p_{k+1}} - n \right) - \sum_{J \in \mathcal{F}_\Theta} \mu_J \left((|J| - 1)n - \sum_J \alpha_{ij} \right) \\ & - \sum_{J \in \mathcal{G}_\Theta} \gamma_J \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) < 0. \end{aligned} \quad (6.65)$$

Since there exists at least one positive term in the two summations, the inequality follows immediately if $\theta_1 - \theta_2 \leq 0$. We may assume $\theta_1 - \theta_2 = 1$ without loss of generality. Write initially given γ_J as $\gamma_J(0)$ for $J \in \mathcal{G}_\Theta$. It is convenient to put $\gamma_J(0) = 0$ for those nonempty subsets $J \notin \mathcal{G}_\Theta$. Then we have

$$\sum_{\Theta} \alpha_{i, k+1} + \frac{n}{p_{k+1}} - n \leq \sum_{J \subset S} \gamma_J(0) \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) \quad (6.66)$$

under the system of equations (6.64). Actually, we may apply the process (6.72) to $\{\gamma_J(0)\}$ for all nonempty $J \subset S$. If $\{\gamma_J(N-1)\}$ is given, we choose two subsets J_1 and J_2 of S with $\gamma_{J_1}(N-1)\gamma_{J_2}(N-1) > 0$ and then put $\{\gamma_J(N)\}$ as in (6.72). Let $\gamma_S^*(N)$ be the maximum of $\gamma_S(N)$ obtained by all possible processes with N continuous steps. We also write $\{\gamma_J^*(N)\}$ to denote one of those $\{\gamma_J(N)\}$ obtaining from $\{\gamma_J(0)\}$ by N continuous steps such that $\gamma_S(N) = \gamma_S^*(N)$. By passing $\{N\}$ to a subsequence $\{N_t\}$, we obtain Cauchy sequences $\{\gamma_J^*(N_t)\}$ for nonempty $J \subset S$. Denote by $\gamma_J^*(\infty)$ the limit of $\gamma_J^*(N_t)$. By a similar argument in §5, we obtain

$$\gamma_S^*(\infty) = 1 + \min_S \pi_i.$$

If $\pi_i > 0$ for some $i \in \Theta$, then there are two possible cases: (a) $\gamma_S^*(\infty) > 1$; (b) $\gamma_S^*(\infty) = 1$ but there exist positive $\gamma_J^*(\infty) > 0$ for those $J \subsetneq S$ containing i . In the case (a), the inequality (6.65) is obvious. In the case (b), we get

$$\begin{aligned} \sum_{\Theta} \alpha_{i, k+1} + \frac{n}{p_{k+1}} - n & < \sum_{J \subset S} \gamma_J^*(\infty) \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) \\ & \leq \sum_{J \in \mathcal{G}_\Theta} \gamma_J(0) \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right). \end{aligned}$$

Then the inequality (6.65) still holds.

Now assume all $\pi_i = 0$ for $i \in \Theta$. If there exists $\mu_J(0) > 0$ for some $J \in \mathcal{F}_\Theta$, then the inequality (6.65) follows from (6.66). Otherwise, all $\mu_J(0) = 0$. The equations (6.64) give

$$\begin{cases} \sum_{J \ni i, j} \gamma_J = 1 + \pi_{ij}, & i < j \in \Theta; \\ \sum_{J \ni i} \gamma_J = 1, & i \in \Theta. \end{cases}$$

This implies that all π_{ij} are zero since π_{ij} are nonnegative. Hence we get $\sum_{\Theta \subset J} \gamma_J(0) = 1$ with the summation taken over $J \subset S$ with $\Theta \subset J$. Then we see that Θ is a proper subset of S and there exists a positive $\gamma_J(0)$ for some $\Theta \subset J \subsetneq S$. Observe that

$$\sum_{\Theta} \alpha_{i,k+1} + \frac{n}{p_{k+1}} - n < |J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i}$$

for all J satisfying $\Theta \subset J \subsetneq S$. Then it follows that

$$\begin{aligned} \sum_{\Theta} \alpha_{i,k+1} + \frac{n}{p_{k+1}} - n &= \sum_{\Theta \subset J \subset S} \gamma_J(0) \left(\sum_{\Theta} \alpha_{i,k+1} + \frac{n}{p_{k+1}} - n \right) \\ &< \sum_{\Theta \subset J \subset S} \gamma_J(0) \left(|J|n - \sum_J \alpha_{ij} - \sum_J \frac{n}{p_i} \right) \end{aligned}$$

which implies the desired inequality.

Case 2: $\sum_{i=2}^k \alpha_{i,k+1} + n/p_{k+1} = n$.

Observe that the existence of solutions to the system (VI.3) is independent of the assumption $\sum_{i=2}^k \alpha_{i,k+1} + n/p_{k+1} < n$. This implies that there also exist solutions when $\sum_{i=2}^k \alpha_{i,k+1} + n/p_{k+1} = n$. Assume $\{\delta_{ij}, \delta_i\}$ is a solution of the system (VI.3). If there exists a positive δ_{ij} , denoted by $\delta_{i_0 j_0}$, then we choose $i_1 \in \Theta - \{1\}$ and put

$$\overline{\alpha_{ij}} = \alpha_{ij} + \delta_{ij} - \delta_i^{i_0} \delta_j^{j_0} \varepsilon, \quad \frac{1}{\overline{p_i}} = \frac{1}{p_i} + \delta_i + \delta_i^{i_1} \varepsilon$$

with $\varepsilon > 0$ sufficiently small. Otherwise, all $\delta_{ij} = 0$ and this implies $\delta_{i_0} > 0$ for some $i_0 \in \Theta$. Choose $i_1 \in \Theta - \{1\}$ arbitrarily. Put

$$\frac{1}{\overline{p_i}} = \frac{1}{p_i} + \delta_i - \delta_i^{i_0} \varepsilon + \delta_i^{i_1} \varepsilon$$

with small $\varepsilon > 0$. Then $\{\overline{\alpha_{ij}}\}$ and $\{\overline{p_i}\}$ are just desired and then the inequality (6.63) is obtained by the induction hypothesis for $k-1$.

Case 3: $\sum_{i=2}^k \alpha_{i,k+1} + n/p_{k+1} > n$.

By Lemma 2.5, \mathbf{L}_1 equals

$$\left(\sum_{\Theta} |x_i - x_j| + \sum_{\Theta} |x_i| \right)^{-\alpha_{1,k+1}} \int_{\mathbb{R}^n} \prod_{\Theta - \{1\}} |x_i - x_{k+1}|^{-\alpha_{i,k+1}} |x_{k+1}|^{-n/p_{k+1}} dx_{k+1}.$$

To use the induction hypothesis, we see that the system to be solved can be given by

$$(VI.4) \begin{cases} (i) & \delta_{ij} \geq 0, \delta_i \geq 0 \quad \text{in } \Theta; \\ (ii) & \sum_{\Theta} \delta_{ij} + n \sum_{\Theta} \delta_i = \alpha_{1,k+1}; \\ (iii) & \sum_{J \cap \Theta} \delta_{ij} < (|J| - 1)n - \sum_J \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right), \quad J \in \overline{\mathcal{F}}_{\Theta}; \\ (iv) & \sum_{J \cap \Theta} \delta_{ij} + n \sum_{J \cap \Theta} \delta_i < |J|n - \sum_J \left(\alpha_{ij} - \delta_i^1 \delta_j^{k+1} \alpha_{1,k+1} \right) - \sum_J n/p_i, \quad J \in \overline{\mathcal{G}}_{\Theta}; \end{cases}$$

where $\overline{\mathcal{F}}_{\Theta}$ consists of all subsets J of $\{1, \dots, k+1\}$ with $|J \cap \Theta| \geq 2$ and $\overline{\mathcal{G}}_{\Theta}$ is the class of all proper subsets J of $\{1, \dots, k+1\}$ with $|J \cap \Theta| \geq 1$. The existence of solutions can be proved

similarly as the system (V.4) in § 5. We omit the details here. Thus we can reduce the desired estimate (6.63) to the case in which the number of $i \in S$ with $\alpha_{i,k+1} > 0$ is equal to $|\Theta| - 1$. Thus the desired inequality holds by induction.

The corresponding estimates involving \mathbf{L}_i with $i \in \Theta$ can be treated similarly as \mathbf{L}_1 . However, we shall treat the estimate related to \mathbf{L}_{k+1} separately. In the case $\sum_{\Theta} \alpha_{i,k+1} \leq n$, the argument is the same as above essentially. Now assume $\sum_{\Theta} \alpha_{i,k+1} > n$. Then \mathbf{L}_{k+1} equals

$$\left(\sum_{\Theta} |x_i - x_j| + \sum_{\Theta} |x_i| \right)^{-n/p_{k+1}} \int_{\mathbb{R}^n} \prod_{\Theta} |x_i - x_{k+1}|^{-\alpha_{i,k+1}} dx_{k+1}.$$

Then the desired system of linear inequalities can be written as follows,

$$(VI.5) \begin{cases} (i) & \delta_{ij} \geq 0, \delta_i \geq 0 \quad \text{in } \Theta; \\ (ii) & \sum_{\Theta} \delta_{ij} + n \sum_{\Theta} \delta_i = \frac{n}{p_{k+1}}; \\ (iii) & \sum_{J \cap \Theta} \delta_{ij} < (|J| - 1)n - \sum_J \alpha_{ij}, \quad J \in \overline{\mathcal{F}}_{\Theta}; \\ (iv) & \sum_{J \cap \Theta} \delta_{ij} + n \sum_{J \cap \Theta} \delta_i < |J|n - \sum_J \alpha_{ij} - \sum_J \left(\frac{n}{p_i} - \delta_i^{k+1} \frac{n}{p_{k+1}} \right), \quad J \in \overline{\mathcal{G}}_{\Theta} \end{cases}$$

where $\overline{\mathcal{F}}_{\Theta}$ and $\overline{\mathcal{G}}_{\Theta}$ are defined as in the system (VI.4). The existence of a solution to the system (VI.5) can be proved similarly as the system (VI.3). Details are omitted here. Assume $\{\delta_{ij}, \delta_i\}$ is a solution to the system (VI.5). Let \overline{p}_i and $\overline{\alpha}_{ij}$ be given by

$$\begin{aligned} \frac{1}{\overline{p}_i} &= \frac{1}{p_i} + \sum_{u \in \Theta} \delta_u^i \delta_u, \quad i \in S; \\ \overline{\alpha}_{ij} &= \alpha_{ij} + \sum_{u < v \in \Theta} \delta_u^i \delta_v^j \delta_{uv}, \quad 1 \leq i < j \leq k+1. \end{aligned}$$

Put $\overline{p}_{k+1} = \infty$. Then $\{\overline{\alpha}_{ij}\}$ and $\{\overline{p}_i\}$ satisfy the assumptions in Theorem 5.4. As a result, we obtain a finite set Δ such that for each $t \in \Delta$ there are nonnegative numbers $\{\beta_{ij}(t) : i < j \in S\}$ and $\{\overline{p}_i : i \in S\}$ satisfying (i), (ii) and (a) of inequalities (iii) in Theorem 1.1 with $k+1$ replaced by k . And also we have

$$\begin{aligned} & |x_1|^{-n\delta_1} \int_{\mathbb{R}^{nk}} \prod_{1 \leq i < j \leq k+1} |x_i - x_j|^{-\overline{\alpha}_{ij}} \prod_{i=2}^{k+1} |x_i|^{-n/\overline{p}_i} dx_2 \cdots dx_{k+1} \\ & \leq C |x_1|^{-n\delta_1} \sum_{t \in \Delta} \int_{\mathbb{R}^{n(k-1)}} \prod_S |x_i - x_j|^{-\beta_{ij}(t)} \prod_{i=2}^k |x_i|^{-n/\overline{p}_i} dx_2 \cdots dx_k \\ & \leq C |x_1|^{-n(1-1/p_1)} \end{aligned}$$

by the induction hypothesis for $k-1$.

Combing above results, we have completed the proof of the theorem. \square

Now we shall give a complete proof of Theorem 1.1 by a multilinear interpolation. We first establish the statement on conditions (i), (ii) and the first type (a) of (iii) in Theorem 1.1. This can be proved similarly as in the case $k=2$ in § 4. By Theorem 6.1, it is easy to find $k+1$ affinely independent points $(1/p_2^{(i)}, \dots, 1/p_{k+1}^{(i)})$ in \mathbb{R}^k for $1 \leq i \leq k+1$ since the additional

assumption (6.56) is satisfied by using (a) of (iii). If there were a nonempty and proper subset J of $S \cup \{k+1\}$ such that $\sum_{i \in J} \sum_{j \in J^c} \alpha_{ij} = 0$, we would see that the homogeneous assumption (i) contradicts the first type of inequalities (iii) for $I = J$ and $I = J^c$. Hence (6.56) is true in the present situation. Without loss of generality, we may further assume $(1/p_1, 1/p_2, \dots, 1/p_{k+1})$ lies in the open convex hull of $k+1$ points $(1/p_1^{(i)}, 1/p_2^{(i)}, \dots, 1/p_{k+1}^{(i)})$. By Theorem 6.2, T is bounded from $L^{p_2^{(i)}} \times \dots \times L^{p_{k+1}^{(i)}}$ into $L^{q_1^{(i)}, \infty}$ with $q_1^{(i)}$ being the conjugate number to $p_1^{(i)}$. Observe that $\sum_{i=2}^{k+1} 1/p_i > 1/p_1'$ by (i) and (ii) in Theorem 1.1. Therefore we may apply Theorem 2.2 to conclude that T is bounded from $L^{p_2} \times \dots \times L^{p_{k+1}}$ into $L^{p_1'}$. Combing the result just obtained together with Theorem 5.4, we also obtain the L^1 estimate in Theorem 1.2 with $p_{k+1} = \infty$.

We now prove the remaining part of Theorem 1.1. Assume there are proper subsets J of $\{1, 2, \dots, k+1\}$ with $|J| \geq 2$ such that $\sum_J 1/p_i + \sum_J \alpha_{ij}/n = |J|$. It is worth noting that the fact $|J| \geq 2$ is implied by $1 < p_i < \infty$. Let $k+1-m$ be the maximum of $|J|$ over all these proper subsets J for some $1 \leq m \leq k-1$. We take a J_0 such that $|J_0|$ attains the maximum $k+1-m$ and $\sum_{J_0} 1/p_i + \sum_{J_0} \alpha_{ij}/n = |J_0|$. By Theorem 3.1, we can reduce matters to two inequalities. First observe that J_0^c contains at least two elements since $\sum_{J_0^c} 1/p_i \geq 1$ by assumption (iii) and $1 < p_i < \infty$. The choice of J_0 implies

$$\sum_J \left(\frac{1}{p_i} + \sum_{j \in J_0} \frac{\alpha_{ij}}{n} \right) + \sum_J \frac{\alpha_{ij}}{n} < |J| \quad (6.67)$$

for all nonempty and proper subsets J of J_0^c . Indeed, if

$$\sum_J \left(\frac{1}{p_i} + \sum_{j \in J_0} \frac{\alpha_{ij}}{n} \right) + \sum_J \frac{\alpha_{ij}}{n} = |J|$$

for some nonempty $J \subsetneq J_0^c$, we obtain

$$\sum_{J \cup J_0} \frac{1}{p_i} + \sum_{J \cup J_0} \frac{\alpha_{ij}}{n} = |J \cup J_0|$$

which contradicts our choice of J_0 since $J \cup J_0$ is also a proper subset of $\{1, 2, \dots, k+1\}$.

Now we claim that the inequality (3.10) is true for the above chosen J_0 . By symmetry, we may assume $J_0 = \{m+1, \dots, k+1\}$ for $2 \leq m \leq k-1$. It is clear that $J_0^c = \{1, 2, \dots, m\}$. To establish (3.10), we introduce a weighted operator $T_w(f_1, \dots, f_{m-1})$ given by

$$T_w(f_1, \dots, f_{m-1})(x_m) = \int_{\mathbb{R}^{n(m-1)}} \prod_{i=1}^{m-1} f_i(x_i) \prod_{1 \leq i < j \leq m} |x_i - x_j|^{-\alpha_{ij}} \prod_{i=1}^m |x_i|^{-\beta_i} dx_1 \cdots dx_{m-1},$$

where $\beta_i = \sum_{j=m+1}^{k+1} \alpha_{ij}$ for each $1 \leq i \leq m$. By Theorem 6.2 and (6.67), we invoke Lemma 2.1 to obtain that the integral of $|T_w(f_1, \dots, f_{m-1})(x_m)g(x_m)|$ over \mathbb{R}^n is bounded by a constant multiple of

$$\left(\prod_{i=1}^{m-1} \|f_i\|_{p_i} \right) \int_{\mathbb{R}^n} T_w(|x_1|^{-n/p_1}, \dots, |x_{m-1}|^{-n/p_{m-1}}) g^*(x_m) dx_m \leq C \left(\prod_{i=1}^{m-1} \|f_i\|_{p_i} \right) \|g\|_{L^{p_m, 1}},$$

for all $g \in L^{p_m, 1}$ with the constant $C < \infty$ not depending on choices of f_i and g . In the hyperplane

$$\sum_{i=1}^m \left(x_i + \beta_i/n \right) + \sum_{1 \leq i < j \leq m} \alpha_{ij}/n = m$$

for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, we see that all points $(1/r_1, \dots, 1/r_m)$ sufficiently close to the fixed one $(1/p_1, \dots, 1/p_m)$ also satisfy the inequalities (6.67). Then we obtain that T_w is bounded from $L^{r_1} \times \dots \times L^{r_{m-1}}$ to $L^{r_m, \infty}$. Choose such m points $(1/r_1^{(i)}, \dots, 1/r_m^{(i)})$ in the above hyperplane, such that $(1/r_1^{(i)}, \dots, 1/r_{m-1}^{(i)})$ are affinely independent in \mathbb{R}^{m-1} and $(1/p_1, \dots, 1/p_m)$ lies in the open convex hull of these points. By the assumption $\sum_{i=1}^m 1/p_i \geq 1$ in Theorem 1.1, we can apply Theorem 2.2 to conclude that T_w is bounded from $L^{p_1} \times \dots \times L^{p_{m-1}}$ into L^{q_m} with q_m being the conjugate exponent to p_m . By duality, we obtain the desired inequality (3.10). The proof of (3.9) is similar as above.

Combing above results, we have completed the proof of Theorem 1.1.

Now we turn to prove that T has a bounded extension from $L^{p_1} \times \dots \times L^{p_k}$ into BMO under the assumptions in Theorem 1.2 with $p_{k+1} = 1$. It is worth noting that we may replace BMO by L^∞ in Theorem 1.2 if (b) of (iii) in Theorem 1.1 is true for $I = S$ there. For $f_i \in C_0^\infty$, Theorem 5.1 suggests that $T(f_1, f_2, \dots, f_k)$ is locally integrable.

For each cube Q with sides parallel to the axes, we use *Q to denote the cube which is concentric to Q but has the side length twice as long as that of Q . We first decompose T , corresponding to Q , as a major term

$$T_S(f_1, \dots, f_k) = \int_{(^*Q^c)^k} \prod_S f_i(x_i) \prod_{S \cup \{k+1\}} |x_i - x_j|^{-\alpha_{ij}} dV_S$$

and k terms of the form

$$T_i(f_1, \dots, f_k) = \int_{^*Q} \left(\int_{(\mathbb{R}^n)^{k-1}} \prod_S f_i(x_i) \prod_{S \cup \{k+1\}} |x_i - x_j|^{-\alpha_{ij}} dV_{S-\{i\}} \right) dx_i$$

for $i \in S$. Here dV_J is the product Lebesgue measure $\prod_{j \in J} dx_j$. As in §4, for $1 \leq i \leq k$ we claim that

$$\int_Q |T_i(f_1, \dots, f_k)(x_{k+1})| dx_{k+1} \leq C|Q| \prod_S \|f_i\|_{p_i}.$$

Take $i = 1$ for example. Put $\overline{p_1} = \left(\frac{\epsilon}{1+\epsilon} + \frac{1}{p_1} \right)^{-1}$, $\overline{p_{k+1}} = 1 + \epsilon$, and $\overline{p_i} = p_i$ for $2 \leq i \leq k$. Thus $\{\overline{p_i}\}$ and $\{\alpha_{ij}\}$ satisfy conditions (i), (ii) and (a) of (iii) in Theorem 1.1 with small $\epsilon > 0$. Thus

$$\begin{aligned} \int_Q |T_1(f_1, \dots, f_k)| dx_{k+1} &\leq \Lambda(|f_1| \chi_{^*Q}, |f_2|, \dots, |f_k|, \chi_Q) \\ &\leq C|Q| \prod_S \|f_i\|_{p_i}. \end{aligned}$$

Similarly, we can show that the estimate still holds for $T_i(f_1, \dots, f_k)$ with $2 \leq i \leq k$.

Now it remains to show the average of $|T_S(f_1, \dots, f_k) - T_S(f_1, \dots, f_k)_Q|$ over Q is bounded by a constant multiple of $\prod_S \|f_i\|_{p_i}$. Let Θ consist of those $i \in S$ such that $\alpha_{i,k+1} > 0$. Then Θ is a nonempty set by assumptions in Theorem 1.2. It is easy to see that

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T_S(f, \dots, f_k) - T_S(f_1, \dots, f_k)_Q| dx_{k+1} \\ &\leq C|Q|^{1/n} \sum_{t \in \Theta} \int_{(^*Q^c)^k} \prod_{i=1}^k |\widetilde{f_i^{(t)}}(x_i)| \prod_{1 \leq i < j \leq k} |x_i - x_j|^{-\alpha_{ij}} dV_k, \end{aligned} \quad (6.68)$$

with

$$\widetilde{f_i^{(t)}}(x_i) = f_i(x_i)|x_i - c_Q|^{-\alpha_{i,k+1}-\delta_i^t}, \quad 1 \leq i \leq k.$$

The estimate of each term in the above summation is similar. To obtain the desired estimate, we shall prove that each term is not greater than a constant multiple of $\prod_{i=1}^k \|f_i\|_{p_i}$. By Theorem 1.1, we need to solve the following system of linear inequalities:

$$(VI.6) \quad \begin{cases} (i) & \delta_i \geq \frac{1}{p_i}, \quad 1 \leq i \leq k; \\ (ii) & \delta_i < 1/p_i + (\alpha_{i,k+1} + \delta_i^t)/n, \quad \text{if } \alpha_{i,k+1} > 0; \\ & \delta_i \leq 1/p_i + \alpha_{i,k+1}/n, \quad \text{if } \alpha_{i,k+1} = 0; \\ (iii) & \sum_{i=1}^k \delta_i = k - \sum_S \alpha_{ij}/n; \\ (iv) & \sum_J \delta_i < |J| - \sum_J \alpha_{ij}/n \quad \text{for nonempty } J \subsetneq S. \end{cases}$$

We first present an useful observation which states that there holds

$$\sum_{i \in J} \sum_{j \in J^c} \alpha_{ij} > 0 \quad (6.69)$$

for all nonempty and proper subsets J of S . Here J^c is the complement of J relative to S . If there were some nonempty $J_1 \subsetneq S$ such that all $\alpha_{ij} = 0$ for $i \in J_1$ and $j \in J_1^c$, we would obtain a contradiction. By assumptions in Theorem 1.2, we have

$$\sum_J \frac{1}{p_i} + \sum_{J \cup \{k+1\}} \frac{\alpha_{ij}}{n} + \frac{1}{p_{k+1}} < |J| + 1$$

with $J = J_1$ and $J = J_1^c$. Recall $p_{k+1} = 1$. The above two inequalities contradict the homogeneous condition (i) in Theorem 1.2. Thus the observation is true.

Suppose that $u_i, v_i, \theta_1, \theta_2$ and μ_J are nonnegative numbers satisfying

$$u_i - v_i + (\theta_1 - \theta_2) - \sum_{J \ni i} \mu_J = 0, \quad 1 \leq i \leq k, \quad (6.70)$$

where either some $v_i > 0$ for those i satisfying $\alpha_{i,k+1} > 0$ or $\mu_J > 0$ for some nonempty proper subset J of S . Our task is to show

$$\begin{aligned} & \sum_S \frac{u_i}{p_i} - \sum_S v_i \left(\frac{1}{p_i} + \frac{\alpha_{i,k+1} + \delta_i^t}{n} \right) + (\theta_1 - \theta_2) \left(k - \sum_S \frac{\alpha_{ij}}{n} \right) - \sum_{J \subset S} \mu_J \left(|J| - \sum_J \frac{\alpha_{ij}}{n} \right) \\ &= - \sum_{J \subset S} \mu_J \left(|J| - \sum_J \frac{1}{p_i} - \sum_J \frac{\alpha_{ij}}{n} \right) + (\theta_1 - \theta_2) B_S - \sum_S v_i \frac{\alpha_{i,k+1}}{n} - \frac{v_t}{n} < 0 \end{aligned} \quad (6.71)$$

where $B_J = |J| - \sum_J 1/p_i - \sum_J \alpha_{ij}/n$ for nonempty subsets J of S . For convenience, we use \mathbf{H} to denote the function in the above inequality. If $\theta_1 - \theta_2 \leq 0$, the desired inequality follows immediately. For $\theta_1 - \theta_2 > 0$, we assume $\theta_1 - \theta_2 = 1$.

Now we also need a process described as in (6.72). For nonempty subsets J_1 and J_2 of S with $\mu_{J_1}(N-1)\mu_{J_2}(N-1) > 0$, put

$$\begin{cases} \mu_{J_t}(N) = \mu_{J_t}(N-1) - \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1), & t = 1, 2. \\ \mu_{J_1 \cap J_2}(N) = \mu_{J_1 \cap J_2}(N-1) + \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) & \text{if } J_1 \cap J_2 \neq \emptyset \\ \mu_{J_1 \cup J_2}(N) = \mu_{J_1 \cup J_2}(N-1) + \mu_{J_1}(N-1) \wedge \mu_{J_2}(N-1) \end{cases} \quad (6.72)$$

and $\mu_J(N) = \mu_J(N-1)$ for other nonempty subsets J of S . Herer we also do not require $J_1 \cap J_2 \neq \emptyset$ in the recursion. For any initial data $\{\mu_J(0)\}$, let $\{\mu_J^*(N)\}$ be obtained by one of those processes consisting of N steps such that $\mu_S(N)$ attains its maximum $\mu_S^*(N)$. Since the process does not change the values of u_i and v_i , we can denote the limit of $\mu_S^*(N)$ by $\mu_S^*(\infty)$ up to a subsequence. By this process, we also have

$$-\sum_{J \subset S} \mu_J(0) \left(|J| - \sum_J \frac{1}{p_i} - \sum_J \frac{\alpha_{ij}}{n} \right) \leq -\sum_{J \subset S} \mu_J^*(\infty) \left(|J| - \sum_J \frac{1}{p_i} - \sum_J \frac{\alpha_{ij}}{n} \right) \quad (6.73)$$

which becomes a strict inequality if $\mu_S^*(\infty) > \mu_S(0)$ by the property (6.69). Let $\omega_i = u_i - v_i$ for $1 \leq i \leq k$. Without loss of generality, assume $\omega_i \geq \omega_{i+1}$ for $1 \leq i \leq k-1$ and

$$\mu_S = 1 + \omega_k, \quad \mu_{\{1,2,\dots,i\}} = \omega_i - \omega_{i+1}, \quad 1 \leq i \leq k-1.$$

Then the object function \mathbf{H} is equal to

$$\mathbf{H} = -\sum_{i=1}^{k-1} (\omega_i - \omega_{i+1}) B_i - \sum_S u_i \frac{\alpha_{i,k+1}}{n} - \frac{v_t}{n}.$$

Since $w_i - w_{i+1} \geq 0$ and $B_i = B_{\{1,\dots,i\}} > 0$ for $1 \leq i \leq k-1$, the desired inequality $\mathbf{H} < 0$ follows if $w_i - w_{i+1} > 0$ for some $1 \leq i \leq k-1$. Assume now $\omega_i = \omega_j$ for all i, j . Then we have $\mathbf{H} = -\sum_S u_i \frac{\alpha_{i,k+1}}{n} - \frac{v_t}{n}$. If $\omega_1 = u_1 - v_1 > 0$, then all u_i are positive and hence $\mathbf{H} < 0$. If $\omega_1 = u_1 - v_1 = 0$, then $u_i = v_i$ for $i \in S$. If there is a positive u_i for those i such that $\alpha_{i,k+1} > 0$, we also get $\mathbf{H} < 0$. Otherwise by the choice of u_i, v_i and μ_J , we see that there exists a $\mu_J(0) > 0$ for a nonempty $J \subsetneq S$. However, 6.73 becomes a strict inequality in this case. Therefore we also obtain the desired statement $\mathbf{H} < 0$.

For each $t \in \Theta$, let $\{\delta_i(t) : 1 \leq i \leq k\}$ be a solution of the system (VI.6). Then by Theorem 1.1 and Hölder's inequality we obtain

$$\begin{aligned} & \frac{1}{|Q|} \int_Q \left| T_S(f, \dots, f_k) - T_S(f_1, \dots, f_k)_Q \right| dx_{k+1} \\ & \leq C |Q|^{1/n} \sum_{t \in \Theta} \prod_S \left\| \widetilde{f_i^{(t)}} \right\|_{\frac{1}{\delta_i(t)}} \\ & \leq C \prod_S \|f_i\|_{p_i}. \end{aligned}$$

This proves that T has a bounded extension from $L^{p_1} \times \dots \times L^{p_k}$ into BMO .

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